Preprint typeset in JHEP style

Feynman Path Integrals: Theory and Applications in the Fields of Quantum Mechanics, Statistical Mechanics and Quantum Field Theory

Lecture notes for the course given in Modena December 2013-January 2014

Olindo Corradini

Centro de Estudios en Física y Matemáticas Basicas y Aplicadas
Universidad Autónoma de Chiapas, Ciudad Universitaria
Tuxtla Gutiérrez 29050, Chiapas, Mexico

Dipartimento di Scienze Fisiche, Informatiche e Matematiche
Università di Modena e Reggio Emilia
Via Campi 213/A, I-41125 Modena, Italy

E-mail: olindo.corradini@unach.mx

Abstract: The following set of lectures cover introductory material on quantum-mechanical Feynman path integrals. We define and apply the path integral to several particle models and quantum field theories in flat space. We start considering the nonrelativistic bosonic particle in a potential for which we compute the exact path integrals for the free particle and for the harmonic oscillator and then consider perturbation theory for an arbitrary potential. We then move to relativistic particles, bosonic and fermionic (spinning) ones, and start considering them from the classical view point studying the symmetries of their actions. We then consider canonical quantization and path integrals and underline the role these models have in the study of space-time higher-spin fields. Finally we generalize the path integral formalism to quantum field theory focalizing on a self-interacting scalar field theory.

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1 Introduction

The notion of *path integral* as integral over trajectories was first introduced by Wiener in the 1920's to solve problems related to the Brownian motion. Later, in 1940's, it was reintroduced by Feynman as an alternative to operatorial methods to compute transition amplitudes in quantum mechanics: Feynman path integrals use a lagrangian formulation instead of a hamiltonian one and can be seen as a quantum-mechanical generalization of the least-action principle (see e.g. [1]).

In electromagnetism the linearity of the Maxwell equations in vacuum allows to formulate the Huygens-Fresnel principle that in turn allows to write the wave scattered by a multiple slit as a sum of waves generated by each slit, where each single wave is characterized by the optical length $I(x_i, x)$ between the i-th slit and the field point x, and the final amplitude is thus given by $A = \sum_i e^{iI(x_i,x)}$, whose squared modulus describes patterns of interference between single waves. In quantum mechanics, a superposition principle can be formulated in strict analogy to electromagnetism and a linear equation of motion, the Schrödinger equation, can be correspondingly postulated. Therefore, the analogy can be carried on further replacing the electromagnetic wave amplitude by a transition amplitude between an initial point x' at time 0 and a final point x at time t, whereas the optical length is replaced by the classical action for going from x' and x in time t (divided by $\hbar$, that has dimensions of action). The full transition amplitude will thus be a "sum" over all paths connecting x' and x in time t:

$$K(x, x'; t) \sim \sum_{\{x(\tau)\}} e^{iS[x(\tau)]/\hbar} \ . \tag{1.1}$$

In the following we try to make sense of the previous expression. Let us now just try to justify the presence of the action in the previous expression by recalling that the action principle applied to the action function $S(x, t)$ (that corresponds to $S[x(\tau)]$ with only the initial point $x' = x(0)$ fixed) implies that the latter satisfy the classical Hamilton-Jacobi equation. Hence the Schrödinger equation imposed onto the amplitude $K(x, x'; t) \sim e^{iS(x,t)/\hbar}$ yields an equation that deviates from Hamilton-Jacobi equation by a linear term in $\hbar$. So that $\hbar$ measures the departure from classical mechanics, that corresponds to $\hbar \to 0$, and the classical action function determines the transition amplitude to leading order in $\hbar$. It will be often useful to parametrize an arbitrary path $x(\tau)$ between x' and x as $x(\tau) = x_{cl}(\tau) + q(\tau)$ with $x_{cl}(\tau)$ being the "classical path" i.e. the path that satisfies the equations of motion with the above boundary conditions, and $q(\tau)$ is an arbitrary deviation (see Figure 1 for graphical description.) With such parametrization we have

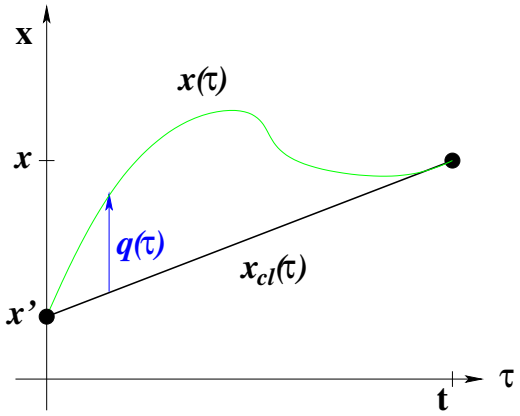


Figure 1. A useful parametrization of paths

$$K(x,x';t) \sim \sum_{\{q(\tau)\}} e^{iS[x_{cl}(\tau)+q(\tau)]/\hbar} \ . \tag{1.2}$$

and $x_{cl}(\tau)$ represents as a sort of "origin" in the space of all paths connecting x' and x in time t.

2 Path integral representation of quantum mechanical transition amplitude: non relativistic bosonic particle

The quantum-mechanical transition amplitude for a time-independent hamiltonian operator is given by (here and henceforth we use natural units and thus set $\hbar = c = 1$; see Appendix A for a brief review on the argument) [1]

$$K(x,x';t) = \langle x|e^{-it\mathbb{H}}|x'\rangle = \langle x,t|x',0\rangle \tag{2.1}$$

$$K(x,x';0) = \delta(x-x') \tag{2.2}$$

and describes the evolution of the wave function from time 0 to time t

$$\psi(x,t) = \int dx' \ K(x,x';t)\psi(x',0) \tag{2.3}$$

and satisfies the Schrödinger equation

$$i\partial_t K(x,x';t) = HK(x,x';t) \tag{2.4}$$

with H being the hamiltonian operator in coordinate representation; for a non-relativistic particle on a line we have $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$. In particular, for a free particle $V = 0$, it is easy to solve the Schrödinger equation (2.4) with boundary conditions (2.2). One obtains

$$K_f(x,x';t) = N_f(t) \ e^{iS_{cl}(x,x';t)} \ , \tag{2.5}$$

[1] A generalization to time-dependent hamiltonian $\mathbb{H}(t)$ can be obtained with the replacement $e^{-it\mathbb{H}} \to \mathcal{T}e^{-i\int_0^t d\tau \mathbb{H}(\tau)}$ and $\mathcal{T}e$ being a time-ordered exponential.

with

$$N_f(t) = \sqrt{\frac{m}{i2\pi t}} \tag{2.6}$$

$$S_{cl}(x, x'; t) \equiv \frac{m(x - x')^2}{2t} \ . \tag{2.7}$$

The latter as we will soon see is the action for the classical path of the free particle.

In order to introduce the path integral we "slice" the evolution operator in (2.1) by defining $\epsilon = t/N$ and insert $N-1$ decompositions of unity in terms of position eigenstates $\mathbb{1} = \int dx|x\rangle\langle x|$; namely

$$K(x, x'; t) = \langle x|e^{-i\epsilon\mathbb{H}}e^{-i\epsilon\mathbb{H}} \cdots e^{-i\epsilon\mathbb{H}}|x'\rangle \tag{2.8}$$

$$= \int \left(\prod_{i=1}^{N-1} dx_i\right) \langle x_N|e^{-i\epsilon\mathbb{H}}|x_{N-1}\rangle \prod_{j=1}^{N-1} \langle x_j|e^{-i\epsilon\mathbb{H}}|x_{j-1}\rangle \tag{2.9}$$

with $x_N \equiv x$ and $x_0 \equiv x'$. We now insert N spectral decomposition of unity in terms of momentum eigenstates $\mathbb{1} = \int \frac{dx}{2\pi}|p\rangle\langle p|$ and get

$$K(x, x'; t) = \int \left(\prod_{i=1}^{N-1} dx_i\right) \left(\prod_{k=1}^{N} \frac{dp_k}{2\pi}\right) \langle x_N|e^{-i\epsilon\mathbb{H}}|p_N\rangle\langle p_N|x_{N-1}\rangle$$
$$\times \prod_{j=1}^{N-1} \langle x_j|e^{-i\epsilon\mathbb{H}}|p_j\rangle\langle p_j|x_{j-1}\rangle. \tag{2.10}$$

For large N, assuming $\mathbb{H} = \mathbb{T} + \mathbb{V} = \frac{1}{2m}\mathbb{p}^2 + V(\mathbb{q})$, we can use the "Trotter formula" (see e.g. [2])

$$\left(e^{-i\epsilon\mathbb{H}}\right)^N = \left(e^{-i\epsilon\mathbb{V}}e^{-i\epsilon\mathbb{T}} + O(1/N^2)\right)^N \approx \left(e^{-i\epsilon\mathbb{V}}e^{-i\epsilon\mathbb{T}}\right)^N \tag{2.11}$$

that allows to replace $e^{-i\epsilon\mathbb{H}}$ with $e^{-i\epsilon\mathbb{V}}e^{-i\epsilon\mathbb{T}}$ in (2.10), so that one gets $\langle x_j|e^{-i\epsilon\mathbb{H}}|p_j\rangle \approx e^{i(x_jp_j - \epsilon H(x_j,p_j))}$, with $\langle x|p\rangle = e^{ixp}$. Hence,

$$K(x, x'; t) = \int \left(\prod_{i=1}^{N-1} dx_i\right) \left(\prod_{k=1}^{N} \frac{dp_k}{2\pi}\right) \exp\left[i\sum_{j=1}^{N} \epsilon\Big(p_j \frac{x_j - x_{j-1}}{\epsilon} - H(x_j, p_j)\Big)\right] \tag{2.12}$$

with

$$H(x_j, p_j) = \frac{1}{2m}p_j^2 + V(x_j) \tag{2.13}$$

the hamiltonian function. In the large N limit we can formally write the latter as

$$K(x, x'; t) = \int_{x(0)=x'}^{x(t)=x} DxDp \exp\left[i\int_0^t d\tau\Big(p\dot{x} - H(x, p)\Big)\right] \tag{2.14}$$

$$Dx \equiv \prod_{0<\tau<t} dx(\tau)\,,\ Dp \equiv \prod_{0<\tau<t} dp(\tau) \tag{2.15}$$

that is referred to as the "phase-space path integral." Alternatively, momenta can be integrated out in (2.10) as they are (analytic continuation of) gaussian integrals. Completing the square one gets

$$K(x,x';t) = \int \left(\prod_{i=1}^{N-1} dx_i\right) \left(\frac{m}{2\pi i\epsilon}\right)^{N/2} \exp\left[i\sum_{j=1}^{N}\epsilon\left(\frac{m}{2}\left(\frac{x_j - x_{j-1}}{\epsilon}\right)^2 - V(x_j)\right)\right] \tag{2.16}$$

that can be formally written as

$$K(x,x';t) = \int_{x(0)=x'}^{x(t)=x} Dx\ e^{iS[x(\tau)]} \tag{2.17}$$

with

$$S[x(\tau)] = \int_0^t d\tau\Big(\frac{m}{2}\dot{x}^2 - V(x(\tau))\Big)\ . \tag{2.18}$$

Expression (2.17) is referred to as "configuration space path integral" and is interpreted as a functional integral over trajectories with boundary condition $x(0) = x'$ and $x(t) = x$.

To date no good definition of path integral measure Dx is known and one has to rely on some regularization methods. For example one may expand paths on a suitable basis to turn the functional integral into a infinite-dimensional Riemann integral. One thus may regularize taking a large (though finite) number of vectors in the basis, perform the integrals and take the limit at the very end. This regularization procedure, as we will see, fixes the path integral up to an overall normalization constant that must be fixed using some consistency conditions. Nevertheless, ratios of path integrals are well-defined objects and turn out to be quite convenient tools in several areas of modern physics. Moreover one may fix –as we do in the next section– the above constant using the simplest possible model (the free particle) and compute other path integrals via their ratios with the free particle path integral.

2.1 Wick rotation to euclidean time: from quantum mechanics to statistical mechanics

As already mentioned path integrals were born in statistical physics. In fact we can easily obtain the particle partition function from (2.17) by, (a) "Wick rotating" time to imaginary time, namely $it \equiv \beta = 1/k\theta$ (where θ is the temperature) and (b) taking the trace

$$\begin{aligned} Z(\beta) = \mathrm{tr}\ e^{-\beta\mathbb{H}} &= \int dx\ \langle x|e^{-\beta\mathbb{H}}|x\rangle = \int dx\ K(x,x,:-i\beta) \\ &= \int_{PBC} Dx\ e^{-S_E[x(\tau)]} \end{aligned} \tag{2.19}$$

where the euclidean action

$$S_E[x(\tau)] = \int_0^\beta d\tau\ \Big(\frac{m}{2}\dot{x}^2 + V(x(\tau))\Big) \tag{2.20}$$

has been obtained by Wick rotating the worldline time $i\tau \to \tau$, and PBC stands for *periodic boundary conditions* and means that the path integral is taken over all closed paths.

2.2 The free particle path integral

We consider the path integral (2.17) for the special case of a free particle, i.e. $V = 0$. For simplicity we consider a particle confined on a line and rescale the worldline time $\tau \to t\tau$ in such a way that free action and boundary conditions turn into

$$S_f[x(\tau)] = \frac{m}{2t}\int_0^1 d\tau\ \dot{x}^2\,, \quad x(0) = x'\,,\ x(1) = x \tag{2.21}$$

where x and x' are points on the line. The free equation of motion is obviously $\ddot{x} = 0$ so that the aforementioned parametrization yields

$$x(\tau) = x_{cl}(\tau) + q(\tau) \tag{2.22}$$

$$x_{cl}(\tau) = x' + (x - x')\tau = x + (x' - x)(1 - \tau)\,, \quad q(0) = q(1) = 0 \tag{2.23}$$

and the above action reads

$$\begin{aligned} S_f[x_{cl} + q] &= \frac{m}{2t}\int_0^1 d\tau\ (\dot{x}_{cl}^2 + \dot{q}^2 + 2\dot{x}_{cl}\dot{q}) = \frac{m}{2t}(x - x')^2 + \frac{m}{2t}\int_0^1 d\tau\ \dot{q}^2 \\ &= S_f[x_{cl}] + S_f[q] \end{aligned} \tag{2.24}$$

and the mixed term identically vanishes due to equation of motion and boundary conditions. The path integral can thus be written as

$$K_f(x, x'; t) = e^{iS_f[x_{cl}]}\int_{q(0)=0}^{q(1)=0} Dq\ e^{iS_f[q(\tau)]} = e^{i\frac{m}{2t}(x-x')^2}\int_{q(0)=0}^{q(1)=0} Dq\ e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} \tag{2.25}$$

so that by comparison with (2.5,2.6,2.7) one can infer that, for the free particle,

$$\int_{q(0)=0}^{q(1)=0} Dq\ e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} = \sqrt{\frac{m}{i2\pi t}}\ . \tag{2.26}$$

The latter results easily generalize to d space dimensions where $\int_{q(0)=0}^{q(1)=0} Dq\ e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} = \left(\frac{m}{i2\pi t}\right)^{d/2}$.

2.2.1 Direct evaluation of path integral normalization

In order to directly compute the path integral normalization one must rely on a regularization scheme that allows to handle the otherwise illdefined measure Dq. One may expand $q(\tau)$ on a basis of functions with Dirichlet boundary condition on the line

$$q(\tau) = \sum_{n=1}^{\infty} q_n \sin(n\pi\tau) \tag{2.27}$$

with q_n arbitrary real coefficients. The measure can be parametrized as

$$Dq \equiv A\prod_{n=1}^{\infty} dq_n a_n \tag{2.28}$$

with A and a_n numerical coefficients that we fix shortly. One possibility, quite popular by string theorists (see e.g. [3]), is to use a gaussian definition for the measure, namely

$$\int \prod_n dq_n a_n \; e^{-||q||^2} = 1, \quad ||q||^2 \equiv \int_0^t d\tau' q^2(\tau') = t \int_0^1 d\tau q^2(\tau) \tag{2.29}$$

so that $a_n = \sqrt{\frac{t}{2\pi}}$. For the path integral normalization one gets

$$\int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} = A \int \prod_{n=1}^{\infty} dq_n \sqrt{\frac{t}{2\pi}} e^{i\frac{m}{4t}\sum_n (\pi n q_n)^2} = A \prod_{n=1}^{\infty} \sqrt{\frac{2i}{m\pi^2}} t \; \frac{1}{n} \tag{2.30}$$

that is thus expressed in terms of an ill-defined infinite product. For such class of infinite products, expressed by $\prod_n an^b$, zeta-function regularization gives the "regularized" value $a^{\zeta(0)} e^{-b\zeta'(0)} = \sqrt{\frac{(2\pi)^b}{a}}$, that specialized to the above free path integral gives

$$\int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} = A \left(\frac{m}{2i}\right)^{1/4} \frac{1}{\sqrt{2t}} = \sqrt{\frac{m}{i2\pi t}} \tag{2.31}$$

provided $A = \left(\frac{2m}{i\pi^2}\right)^{1/4}$: zeta-function regularization provides the correct functional form (in terms of t) for the path integral normalization.

A different normalization can be obtained by asking instead that each mode is normalized with respect to the free kinetic action, namely

$$\int dq_n a_n e^{i\frac{m}{4t}\sum_n (\pi n q_n)^2} = 1 \;\Rightarrow\; a_n = \sqrt{\frac{mn^2\pi}{4it}} \tag{2.32}$$

and therefore the overall normalization must be fixed as $A = \sqrt{\frac{m}{i2\pi t}}$. The latter normalization is quite useful if used with *mode regularization* where the product (2.28) is truncated to a large finite mode M. This method can be employed to compute more generic particle path integrals where interaction terms may introduce computational ambiguities. Namely: whenever an ambiguity appears one can always rely on the mode expansion, truncated at M, and then take the large limit at the very end, after having resolved the ambiguity. Other regularization schemes that have been adopted to such purpose are: *time slicing* that rely on the well-defined expression for the path integral as multiple time slices (cfr. eq. (2.16)) and *dimensional regularization* that regulates ambiguities by dimensionally extending the worldline (see [4] for a review on such issues). However dimensional regularization is a regularization that only works in the perturbative approach to the path integral, by regulating single Feynman diagrams. Perturbation theory within the free particle path integral is a subject that will be discussed below.

2.2.2 The free particle partition function

The partition function for a free particle in d-dimensional space can be obtained as

$$Z_f(\beta) = \int d^d x \; K(x, x, : -i\beta) = \left(\frac{m}{2\pi\beta}\right)^{d/2} \int d^d x = \mathcal{V} \left(\frac{m}{2\pi\beta}\right)^{d/2} \tag{2.33}$$

$\mathcal{V}$ being the spatial volume. It is easy to recall that this is the correct result as

$$Z_f(\beta) = \sum_p \; e^{-\beta p^2/2m} = \frac{\mathcal{V}}{(2\pi)^d} \int d^d p \; e^{-\beta p^2/2m} = \mathcal{V} \left(\frac{m}{2\pi\beta} \right)^{d/2} \tag{2.34}$$

with $\frac{\mathcal{V}}{(2\pi)^d}$ being the "density of states".

2.2.3 Perturbation theory about free particle solution: Feynman diagrams

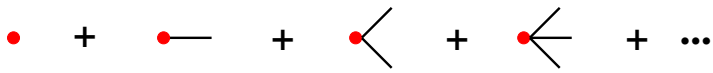
In the presence of an arbitrary potential the path integral for the transition amplitude is not exactly solvable. However if the potential is "small" compared to the kinetic term one can rely on perturbation theory about the free particle solution. The significance of being "small" will be clarified a posteriori.

Let us then obtain a perturbative expansion for the transition amplitude (2.17) with action (2.18). As done above we split the arbitrary path in terms of the classical path (with respect to the free action) and deviation $q(\tau)$ and again making use of the rescaled time we can rewrite the amplitude as

$$K(x, x'; t) = N_f(t) \; e^{i\frac{m}{2t}(x-x')^2} \; \frac{\displaystyle\int_{q(0)=0}^{q(1)=0} Dq \; e^{i\int_0^1 (\frac{m}{2t}\dot{q}^2 - tV(x_{cl}+q))}}{\displaystyle\int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2}} \; . \tag{2.35}$$

We then Taylor expand the potential in the exponent about the classical free solution: this gives rise to a infinite set of interaction terms (τ dependence in x_{cl} and q is left implied)

$$S_{int} = -t \int_0^1 d\tau \Big(V(x_{cl}) + V'(x_{cl})q + \frac{1}{2!}V^{(2)}(x_{cl})q^2 + \frac{1}{3!}V^{(2)}(x_{cl})q^3 + \cdots \Big)$$



(2.36)

Next we expand the exponent $e^{iS_{int}}$ so that we only have polynomials in q to integrate with the path integral weight; in other words we only need to compute expressions like

$$\frac{\displaystyle\int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2} q(\tau_1)q(\tau_2)\cdots q(\tau_n)}{\displaystyle\int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2}} \equiv \Big\langle q(\tau_1)q(\tau_2)\cdots q(\tau_n) \Big\rangle \tag{2.37}$$

and the full (perturbative) path integral can be compactly written as

$$K(x, x'; t) = N_f(t) \; e^{i\frac{m}{2t}(x-x')^2} \Big\langle e^{-it\int_0^1 V(x_{cl}+q)} \Big\rangle \tag{2.38}$$

and the expressions $\Big\langle f(q) \Big\rangle$ are referred to as "correlation functions". In order to compute the above correlations functions we define and compute the so-called "generating functional"

$$\mathcal{Z}[j] \equiv \int_{q(0)=0}^{q(1)=0} Dq \; e^{i\frac{m}{2t}\int_0^1 \dot{q}^2 + i\int_0^1 qj} = N_f(t) \Big\langle e^{i\int_0^1 qj} \Big\rangle \tag{2.39}$$

in terms of which

$$\Big\langle q(\tau_1)q(\tau_2)\cdots q(\tau_n)\Big\rangle = (-i)^n \frac{1}{\mathcal{Z}[0]}\frac{\delta^n}{\delta j(\tau_1)\delta j(\tau_2)\cdots\delta j(\tau_n)}\mathcal{Z}[j]\Big|_{j=0} \,. \tag{2.40}$$

By partially integrating the kinetic term and completing the square we get

$$\mathcal{Z}[j] = e^{\frac{i}{2}\iint jD^{-1}j}\int_{q(0)=0}^{q(1)=0} Dq\; e^{i\frac{m}{2t}\int_0^1 \dot{\tilde{q}}^2} \tag{2.41}$$

where $D^{-1}(\tau,\tau')$, "the propagator", is the inverse kinetic operator, with $D \equiv \frac{m}{t}\partial_\tau^2$, such that

$$DD^{-1}(\tau,\tau') = \delta(\tau,\tau') \tag{2.42}$$

in the basis of function with Dirichlet boundary conditions, and $\tilde{q}(\tau) \equiv q(\tau) - \int_0^1 j(\tau')D^{-1}(\tau',\tau)$. By defining $D^{-1}(\tau,\tau') = \frac{t}{m}\Delta(\tau,\tau')$ we get

$$\dot{\dot{\Delta}}(\tau,\tau') = \delta(\tau,\tau') \tag{2.43}$$

$$\Rightarrow \quad \Delta(\tau,\tau') = \theta(\tau-\tau')(\tau-1)\tau' + \theta(\tau'-\tau)(\tau'-1)\tau \tag{2.44}$$

where "dot" on the left (right) means derivative with respect to τ (τ'). The propagator satisfies the following properties

$$\Delta(\tau,\tau') = \Delta(\tau',\tau) \tag{2.45}$$

$$\Delta(\tau,0) = \Delta(\tau,1) = 0 \tag{2.46}$$

from which $\tilde{q}(0) = \tilde{q}(1) = 0$. Therefore we can shift the integration variable in (2.41) from q to $\tilde{q}$ and get

$$\mathcal{Z}[j] = N_f(t)\; e^{\frac{it}{2m}\iint j\Delta j} \tag{2.47}$$

and finally obtain

$$\Big\langle q(\tau_1)q(\tau_2)\cdots q(\tau_n)\Big\rangle = (-i)^n \frac{\delta^n}{\delta j(\tau_1)\delta j(\tau_2)\cdots\delta j(\tau_n)} e^{\frac{it}{2m}\iint j\Delta j}\Big|_{j=0} \,. \tag{2.48}$$

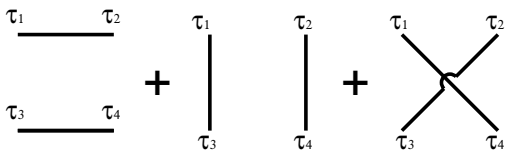
In particular:

1. correlation functions of an odd number of "fields" vanish;

2. the 2-point function is nothing but the propagator

$$\Big\langle q(\tau_1)q(\tau_2)\Big\rangle = -i\frac{t}{m}\Delta(\tau_1,\tau_2) = \tau_1 \text{———} \tau_2 \tag{2.49}$$

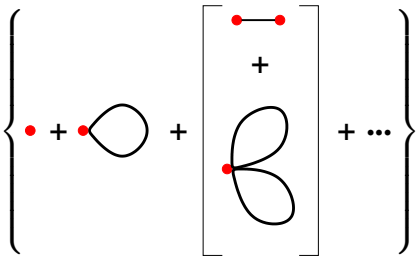
3. correlation functions of an even number of fields are obtained by all possible contractions of pairs of fields. For example, for $n=4$ we have $\langle q_1q_2q_3q_4\rangle = (-i\frac{t}{m})^2(\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23})$ where an obvious shortcut notation has been used. The latter statement is known as the "Wick theorem". Diagrammatically

$\Big\langle q_1q_2q_3q_4\Big\rangle =$ 

Noting that each vertex and each propagator carry a power of t/m we can write the perturbative expansion as a short-time expansion (or inverse-mass expansion). It is thus not difficult to convince oneself that the expansion reorganizes as

$$K(x,x';t) = N_f(t)\; e^{i\frac{m}{2t}(x-x')^2} \exp\left\{\text{connected diagrams}\right\}$$

$$= N_f(t)\; e^{i\frac{m}{2t}(x-x')^2} \exp\Bigg\{$$



$$\Bigg\} \tag{2.50}$$

where the diagrammatic expansion in the exponent (that is ordered by increasing powers of t/m) only involves connected diagrams, i.e. diagrams whose vertices are connected by at least one propagator. We recall that $N_f(t) = \sqrt{\frac{m}{i2\pi t}}$, and we can also give yet another representation for the transition amplitude, the so-called "heat-kernel expansion"

$$K(x,x';t) = \sqrt{\frac{m}{i2\pi t}}\; e^{i\frac{m}{2t}(x-x')^2} \sum_{n=0}^{\infty} a_n(x,x')t^n \tag{2.51}$$

where the terms $a_n(x,x')$ are known as Seeley-DeWitt coefficients. We can thus express such coefficients in terms of Feynman diagrams and get

$$a_0(x,x') = 1 \tag{2.52}$$

$$a_1(x,x') = \bullet = -i\int_0^1 d\tau\; V(x_{cl}(\tau)) \tag{2.53}$$

$$a_2(x,x') = \frac{1}{2!}\langle\bullet\rangle^2 +$$

$$= \frac{1}{2!}\left(-i\int_0^1 d\tau\; V(x_{cl}(\tau))\right)^2 - \frac{1}{2!\,m}\int_0^1 d\tau\; V^{(2)}(x_{cl}(\tau))\Delta(\tau,\tau)$$

$$= \frac{1}{2!}\left(-i\int_0^1 d\tau\; V(x_{cl}(\tau))\right)^2 - \frac{1}{2!\,m}\int_0^1 d\tau\; V^{(2)}(x_{cl}(\tau))\tau(\tau-1) \tag{2.54}$$

.

.

where in (2.54) we have used that $\langle q(\tau)q(\tau)\rangle = -i\frac{t}{m}\Delta(\tau,\tau) = -i\frac{t}{m}\tau(\tau-1)$. Let us now comment on the validity of the expansion: each propagator inserts a power of t/m. Therefore for a fixed potential V, the larger the mass, the larger the time for which the expansion is accurate. In other words for a very massive particle it results quite costy to move away from the classical path.

2.3 The harmonic oscillator path integral

If the particle is subject to a harmonic potential $V_h(x) = \frac{1}{2}m\omega^2x^2$ the path integral is again exactly solvable. In the rescaled time adopted above the path integral can be formally

written as in (2.17) with action

$$S_h[x(\tau)] = \frac{m}{2t}\int_0^1 d\tau\Big[\dot{x}^2 - (\omega t)^2 x^2\Big] \ . \tag{2.55}$$

Again we parametrize $x(\tau) = x_{cl}(\tau) + q(\tau)$ with

$$\left.\begin{array}{l} \ddot{x}_{cl} + (\omega t)^2 x_{cl} = 0 \\ x_{cl}(0) = x' \ , \ x_{cl}(1) = x \\ (q(0) = q(1) = 0) \end{array}\right\} \ \Rightarrow \ x_{cl}(\tau) = x'\cos(\omega t\tau) + \frac{x - x'\cos(\omega t)}{\sin(\omega t)}\sin(\omega t\tau) \tag{2.56}$$

and, since the action is quadratic, similarly to the free particle case there is no mixed term between $x_{cl}(\tau)$ and $q(\tau)$, i.e. $S_h[x(\tau)] = S_h[x_{cl}(\tau)] + S_h[q(\tau)]$ and the path integral reads

$$K_h(x, x'; t) = N_h(t)\ e^{iS_h[x_{cl}]} \tag{2.57}$$

with

$$S_h[x_{cl}] = \frac{m\omega}{2\sin(\omega t)}\Big((x^2 + x'^2)\cos(\omega t) - 2xx'\Big) \tag{2.58}$$

$$N_h(t) = \int_{q(0)=0}^{q(1)=0} Dq\ e^{i\frac{m}{2t}\int_0^1 d\tau\left(\dot{q}^2 - (\omega t)^2 q^2\right)} \ . \tag{2.59}$$

We now use the above mode expansion to compute the latter:

$$\begin{aligned} N_h(t) = N_f(t)\ \frac{N_h(t)}{N_f(t)} &= \sqrt{\frac{m}{i2\pi t}}\ \frac{\displaystyle\int_{q(0)=0}^{q(1)=0} Dq\ e^{i\frac{m}{2t}\int_0^1(\dot{q}^2 - (\omega t)^2 q^2)}}{\displaystyle\int_{q(0)=0}^{q(1)=0} Dq\ e^{i\frac{m}{2t}\int_0^1 \dot{q}^2}} \\ &= \sqrt{\frac{m}{i2\pi t}}\prod_{n=1}^{\infty}\frac{\displaystyle\int dq_n\ e^{i\frac{imt}{4}\sum_n(\omega_n^2 - \omega^2)q_n^2}}{\displaystyle\int dq_n\ e^{i\frac{imt}{4}\sum_n \omega_n^2 q_n^2}} = \sqrt{\frac{m}{i2\pi t}}\prod_{n=1}^{\infty}\left(1 - \frac{\omega^2}{\omega_n^2}\right)^{-1/2} \\ &= \sqrt{\frac{m}{i2\pi t}}\left(\frac{\omega t}{\sin(\omega t)}\right)^{1/2} = \left(\frac{m\omega}{i2\pi\sin(\omega t)}\right)^{1/2} \end{aligned} \tag{2.60}$$

Above $\omega_n \equiv \frac{\pi n}{t}$. It is not difficult to check that, with the previous expression for $N_h(t)$, path integral (2.57) satisfies the Schrödinger equation with hamiltonian $H = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$ and boundary condition $K(x, x'; 0) = \delta(x - x')$. The propagator can also be easily generalized

$$\begin{aligned} \Delta_\omega(\tau, \tau') = \frac{1}{\omega t\sin(\omega t)}\Big\{&\theta(\tau - \tau')\sin(\omega t(\tau - 1))\sin(\omega t\tau') \\ &+\theta(\tau' - \tau)\sin(\omega t(\tau' - 1))\sin(\omega t\tau)\Big\} \end{aligned} \tag{2.61}$$

and can be used in the perturbative approach, for a harmonic oscillator in a potential V. Expression (2.61) is solution of the Green's equation

$$\Big(\partial_\tau^2 + (\omega t)^2\Big)\Delta_\omega(\tau, \tau') = \delta(\tau - \tau') \ . \tag{2.62}$$

In the limit $t \to 0$ (or $\omega \to 0$), all previous expressions reduce to their free particle counterparts.

2.3.1 The harmonic oscillator partition function

Similarly to the free particle case we can switch to statistical mechanics by Wick rotating the time, $it = \beta$ and get

$$K(x, x'; -i\beta) = \langle x|e^{-\beta H}|x'\rangle = e^{-S_h[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq\ e^{-S_h[q]} \tag{2.63}$$

where now

$$S_h[x] = \frac{m}{2\beta} \int_0^1 d\tau \Big(\dot{x}^2 + (\omega\beta)^2 x^2\Big) \tag{2.64}$$

is the euclidean action and

$$S_h[x_{cl}] = \frac{m\omega}{2\sinh(\beta\omega)} \Big[(x^2 + x'^2)\cosh(\beta\omega) - 2xx'\Big] \tag{2.65}$$

$$\int_{q(0)=0}^{q(1)=0} Dq\ e^{-S_h[q]} = \left(\frac{m\omega}{2\pi\sinh(\beta\omega)}\right)^{1/2} \tag{2.66}$$

that is always regular. The classical path, that satisfies $\ddot{x} = (\omega\beta)^2 x$, can also be easily obtained by analytic continuation and reads

$$x_{cl}(\tau) = x'\cosh(\omega\beta\tau) + \frac{x - x'\cosh(\omega\beta)}{\sinh(\omega\beta)}\sinh(\omega\beta\tau)\ . \tag{2.67}$$

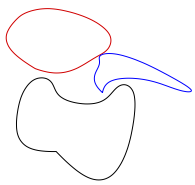
In the zero temperature limit ($\beta \to \infty$) we thus get

$$\langle x|e^{-\beta H}|x'\rangle \ \longrightarrow\ \left(\frac{m\omega}{\pi}\right)^{1/2} e^{-\frac{m\omega}{2}(x^2+x'^2)}\ e^{-\beta\omega/2} = \psi_0(x)\psi_0^*(x')e^{-\beta E_0} \tag{2.68}$$

i.e. the limit correctly singles out the vacuum state.

Taking the trace of the amplitude (heat kernel) one gets the partition function

$$Z_h(\beta) = \int dx\ K_h(x, x; -i\beta) = \int_{PBC} Dx\ e^{-S_h[x]} = \sum$$



$$= \frac{1}{\sqrt{2}} \frac{1}{(\cosh(\beta\omega) - 1)^{1/2}} = \frac{e^{-\frac{\beta\omega}{2}}}{1 - e^{-\beta\omega}} \tag{2.69}$$

where PBC stands for "periodic boundary conditions" $x(0) = x(1)$ and implies that the path integral is a "sum" over all closed trajectories. Above we used the fact that the partition function for the free particle can be obtained either using a transition amplitude computed with Dirichlet boundary conditions $x(0) = x(1) = x$ and integrating the over the initial=final point x, or with periodic boundary conditions $x(0) = x(1)$, integrating over the "center of mass" of the path $x_0 \equiv \int_0^1 d\tau x(\tau)$.

It is easy to check that the latter matches the geometric series $\sum_{n=0}^{\infty} e^{-\beta\omega(n+\frac{1}{2})}$. In particular, taking the zero-temperature limit ($\beta \to \infty$), the latter singles out the vacuum

energy $Z_h(\beta) \approx e^{-\beta\frac{\omega}{2}}$, so that for generic particle models, the computation of the above path integral can be seen as a method to obtain an estimate of the vacuum energy. For example, in the case of an anharmonic oscillator, if the deviation from harmonicity is small, perturbation theory can be used to compute the correction to the vacuum energy.

2.3.2 Perturbation theory about the harmonic oscillator partition function solution

Perturbation theory about the harmonic oscillator partition function solution goes essentially the same way as we did for the free particle transition amplitude, except that now we may use periodic boundary conditions for the quantum fields rather than Dirichlet boundary conditions. Of course one can keep using DBC, factor out the classical solution, with $x_{cl}(0) = x_{cl}(1) = x$ and integrate over x. However for completeness let us choose the former parametrization and let us focus on the case where the interacting action is polynomial $S_{int}[x] = \beta \int_0^1 d\tau \sum_{n>2} \frac{g_n}{n!} x^n$; we can define the generating functional

$$\mathcal{Z}[j] = \int_{PBC} Dx \; e^{-S_h[x] + \int_0^1 jx} \tag{2.70}$$

that similarly to the free particle case yields

$$\mathcal{Z}[j] = Z_h(\beta) \; e^{\frac{1}{2}\iint j D_h^{-1} j} \tag{2.71}$$

and the propagator results

$$\begin{aligned} \Big\langle x(\tau)x(\tau') \Big\rangle &= D_h^{-1}(\tau - \tau') \equiv G_\omega(\tau - \tau') \\ \frac{m}{\beta}(-\partial_\tau^2 + (\omega\beta)^2) G_\omega(\tau - \tau') &= \delta(\tau - \tau') \end{aligned} \tag{2.72}$$

in the space of functions with periodicity $\tau \cong \tau + 1$. On the infinite line the latter equation can be easily inverted using the Fourier transformation that yields

$$G_\omega^l(u) = \frac{1}{2m\omega} e^{-\beta\omega|u|} \; , \quad u \equiv \tau - \tau' \; . \tag{2.73}$$

In order to get to Green's function on the circle (i.e. with periodicity $\tau \cong \tau + 1$) we need to render (2.73) periodic [5]. Using Fourier analysis in (2.72) we get

$$G_\omega(u) = \frac{\beta}{m} \sum_{k\in\mathbb{Z}} \frac{1}{(\beta\omega)^2 + (2\pi k)^2} e^{i2\pi k u} = \frac{\beta}{m} \int_{-\infty}^{\infty} d\lambda \sum_{k\in\mathbb{Z}} \delta(\lambda - k) \frac{1}{(\beta\omega)^2 + (2\pi\lambda)^2} e^{i2\pi\lambda u} \tag{2.74}$$

where in the second passage we inserted an auxiliary integral. Now we can use the Poisson resummation formula $\sum_k f(k) = \sum_n \hat{f}(n)$ where $\hat{f}(\nu)$ is the Fourier transform of $f(x)$, with ν being the frequency, i.e. $\hat{f}(\nu) = \int dx f(x) e^{-i2\pi\nu x}$. In the above case $f(x) = \delta(\lambda - x)$ so that $\hat{f}(n) = e^{-i2\pi\lambda n}$. Hence the leftover integral over λ yields

$$G_\omega(u) = \sum_{n\in\mathbb{Z}} G_\omega^l(u + n) \tag{2.75}$$

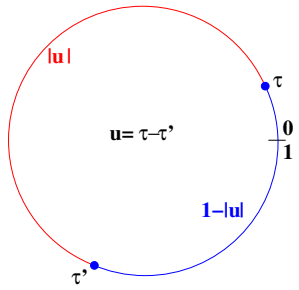


Figure 2. Shortest distance on the circle between τ and τ'

that is explicitly periodic. The latter sum involves simple geometric series that can be summed to give

$$G_\omega(u) = \frac{1}{2m\omega} \frac{\cosh\left(\omega\beta\left(\frac{1}{2} - |u|\right)\right)}{\sinh\left(\frac{\omega\beta}{2}\right)} \ . \tag{2.76}$$

This is the Green's function for the harmonic oscillator with periodic boundary conditions (on the circle). Notice that in the large β limit one gets an expression that is slightly different from the Green's function on the line (cfr. eq.(2.73)), namely:

$$G_\omega^\infty(u) = \frac{1}{2m\omega} \begin{cases} e^{-\beta\omega|u|} \ , & |u| < \frac{1}{2} \\ e^{-\beta\omega(1-|u|)} \ , & \frac{1}{2} < |u| < 1 \ . \end{cases} \tag{2.77}$$

Basically the Green's function is the exponential of the shortest distance (on the circle) between τ and τ', see Figure 2. The partition function for the anharmonic oscillator can be formally written as

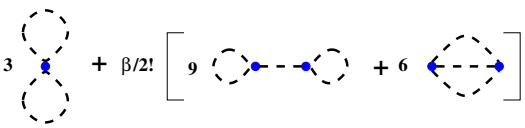
$$\begin{aligned} Z_{ah}(\beta) &= Z_h(\beta)\ e^{-S_{int}[\delta/\delta j]}\ e^{\frac{1}{2}\iint j G_\omega j}\Big|_{j=0} \\ &= Z_h(\beta)\ e^{\{\text{connected diagrams}\}} \ . \end{aligned} \tag{2.78}$$

As an example let us consider the case

$$S_{int}[x] = \beta \int_0^1 d\tau \left(\frac{g}{3!}x^3 + \frac{\lambda}{4!}x^4\right) = \beta \Bigg(\tag{2.79}$$

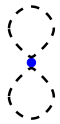
so that, to lowest order

$$Z_{ah}(\beta) = Z_h(\beta) \exp\left\{ \beta \Bigg(- 3 \right. \tag{2.80}$$

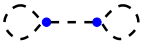


$$\left. \Bigg) \right\}$$

and the single diagrams read



$$= \frac{\lambda}{4!}\int_0^1 d\tau\,\left(G_\omega(0)\right)^2 \overset{\beta\to\infty}{\longrightarrow} = \frac{\lambda}{4\cdot 4!(m\omega)^2} \tag{2.81}$$



$$= \left(\frac{g}{3!}\right)^2 \int_0^1\int_0^1 \left(G_\omega(0)\right)^2 G_\omega(\tau-\tau') \overset{\beta\to\infty}{\longrightarrow} = \frac{g^2}{4(3!)^2\beta m^3\omega^4} \tag{2.82}$$

$$= \left(\frac{g}{3!}\right)^2 \int_0^1\int_0^1 \left(G_\omega(\tau-\tau')\right)^3 \overset{\beta\to\infty}{\longrightarrow} = \frac{g^2}{12(3!)^2\beta m^3\omega^4}\,. \tag{2.83}$$

Then in the zero-temperature limit

$$Z_{ah}(\beta) \approx e^{-\beta E_0'} \tag{2.84}$$

$$E_0' = \frac{\omega}{2}\left(1 + \frac{\lambda}{16m^2\omega^3} - \frac{11g^2}{144m^3\omega^5}\right) \tag{2.85}$$

gives the sought estimate for the vacuum energy. However, the above expression (2.80) with diagrams (2.81,2.82,2.83) computed with the finite temperature Green's function (2.76) yields (the perturbative expansion for) the finite temperature partition function of the anharmonic oscillator described by (2.79).

2.4 Problems for Section 2

(1) Show that the classical action for the free particle on the line is $S_f[x_{cl}] = \frac{m(x-x')^2}{2t}$.

(2) Compute the 6-point correlation function.

(3) Compute the Seeley-DeWitt coefficient $a_3(x,x')$ both diagrammatically and in terms of vertex functions.

(4) Compute the classical action (2.58) for the harmonic oscillator on the line, by replacing the solution (2.56) into the action (2.55).

(5) Show that the transition amplitude (2.57) satisfies the Schrödinger equation for the harmonic oscillator.

(6) Show that the propagator (2.61) satisfies the Green equation $(\partial_\tau^2 + (\omega t)^2)\Delta_\omega(\tau,\tau') = \delta(\tau-\tau')$.

(7) Show that the propagator (2.73) satisfies eq. (2.72).

(8) Using Fourier transformation, derive (2.73) from (2.72).

(9) Using the geometric series obtain (2.76) from (2.75) .

(10) Check that, in the large β limit up to exponentially decreasing terms, expressions (2.81,2.82,2.83) give the same results, both with the Green's function (2.73) and with (2.77).

(11) Compute the next-order correction to the vacuum energy (2.85).

(12) Compute the finite temperature partition function for the anharmonic oscillator given by (2.79) to leading order in perturbation theory, i.e. only consider the eight-shaped diagram.

3 Path integral representation of quantum mechanical transition amplitude: fermionic degrees of freedom

We employ the coherent state approach to generalize the path integral to transition amplitude of models with fermionic degrees of freedom. The simplest fermionic system is a two-dimensional Hilbert space, representation of the anticommutators algebra

$$\{a, a^\dagger\} = 1\,, \quad a^2 = (a^\dagger)^2 = 0\ . \tag{3.1}$$

The spin basis for such algebra is given by $(|-\rangle, |+\rangle)$ where

$$a|-\rangle = 0\,, \quad |+\rangle = a^\dagger|-\rangle\,, \quad |-\rangle = a|+\rangle \tag{3.2}$$

and a spin state is thus a two-dimensional object (a spinor) in such a basis. An alternative, overcomplete, basis for spin states is the so-called "coherent state basis" that, for the previous simple system, is simply given by the following bra's and ket's

$$\begin{aligned} |\xi\rangle &= e^{a^\dagger \xi}|0\rangle = (1 + a^\dagger \xi)|0\rangle \quad \to \quad a|\xi\rangle = \xi|\xi\rangle \\ \langle\bar\eta| &= \langle 0|e^{\bar\eta a} = \langle 0|(1 + \bar\eta a) \quad \to \quad \langle\bar\eta|a^\dagger = \langle\bar\eta|\bar\eta \end{aligned} \tag{3.3}$$

and can be generalized to an arbitrary set of pairs of fermionic generators; see Appendix C for details. Coherent states (3.3) satisfy the following properties

$$\langle\bar\eta|\xi\rangle = e^{\bar\eta\xi} \tag{3.4}$$

$$\int d\bar\eta d\xi\ e^{-\bar\eta\xi}|\xi\rangle\langle\bar\eta| = \mathbb{1} \tag{3.5}$$

$$\int d\xi\ e^{(\bar\lambda-\bar\eta)\xi} = \delta(\bar\lambda - \bar\eta) \tag{3.6}$$

$$\int d\bar\eta\ e^{\bar\eta(\rho\xi)} = \delta(\rho - \xi) \tag{3.7}$$

$$\mathrm{tr}A = \int d\bar\eta d\xi\ e^{-\bar\eta\xi}\langle -\bar\eta|A|\xi\rangle = \int d\xi d\bar\eta\ e^{\bar\eta\xi}\langle\bar\eta|A|\xi\rangle\ . \tag{3.8}$$

Let us take $|\phi\rangle$ as initial state, then the evolved state will be

$$|\phi(t)\rangle = e^{-it\mathbb{H}}|\phi\rangle \tag{3.9}$$

that in the coherent state representation becomes

$$\phi(\bar\lambda; t) \equiv \langle\bar\lambda|\phi(t)\rangle = \langle\bar\lambda|e^{-it\mathbb{H}}|\phi\rangle = \int d\bar\eta d\eta e^{-\bar\eta\eta}\langle\bar\lambda|e^{-it\mathbb{H}}|\eta\rangle\phi(\bar\eta; 0) \tag{3.10}$$

where in the last equality we have used property (3.5). The integrand $\langle\bar{\lambda}|e^{-it\mathbb{H}}|\eta\rangle$ in (3.10) assumes the form of a transition amplitude as in the bosonic case. It is thus possible to represent it with a fermionic path integral. In order to do that let us first take the trivial case $\mathbb{H}=0$ and insert N decompositions of identity $\int d\bar{\xi}_i d\xi_i\ e^{-\bar{\xi}_i\xi_i}|\xi_i\rangle\langle\bar{\xi}_i|=\mathbb{1}$. We thus get

$$\langle\bar{\lambda}|\eta\rangle=\int\prod_{i=1}^{N}d\bar{\xi}_i d\xi_i\ \exp\left[\bar{\lambda}\xi_N-\sum_{i=1}^{N}\bar{\xi}_i(\xi_i-\xi_{i-1})\right],\quad \bar{\xi}_N\equiv\bar{\lambda}\,,\ \xi_0\equiv\eta \tag{3.11}$$

that in the large N limit can be written as

$$\langle\bar{\lambda}|\eta\rangle=\int_{\xi(0)=\eta}^{\bar{\xi}(1)=\bar{\lambda}}D\bar{\xi}D\xi\ e^{iS[\xi,\bar{\xi}]} \tag{3.12}$$

with

$$S[\xi,\bar{\xi}]=i\left(\int_0^1 d\tau\bar{\xi}\dot{\xi}(\tau)-\bar{\xi}\xi(1)\right)\ . \tag{3.13}$$

In the presence of a nontrivial hamiltonian $\mathbb{H}$ the latter becomes

$$\langle\bar{\lambda}|e^{-it\mathbb{H}}|\eta\rangle=\int_{\xi(0)=\eta}^{\bar{\xi}(1)=\bar{\lambda}}D\bar{\xi}D\xi\ e^{iS[\xi,\bar{\xi}]} \tag{3.14}$$

$$S[\xi,\bar{\xi}]=\int_0^1 d\tau\left(i\bar{\xi}\dot{\xi}(\tau)-H(\xi,\bar{\xi})\right)-i\bar{\xi}\xi(1) \tag{3.15}$$

that is the path integral representation of the fermionic transition amplitude. Here a few comments are in order: (a) the fermionic path integral resembles more a bosonic phase space path integral than a configuration space one. (b) The boundary term $\bar{\xi}\xi(1)$, that naturally comes out from the previous construction, plays a role when extremizing the action to get the equations of motion; namely, it cancels another boundary term that comes out from partial integration. It also plays a role when computing the trace of an operator: see below. (c) The generalization from the above naive case to (3.14) is not a priori trivial, because of ordering problems. In fact $\mathbb{H}$ may involve mixing terms between a and $a^\dagger$. However result (3.14) is guaranteed in that form (i.e. the quantum $\mathbb{H}(a,a^\dagger)$ is replaced by $H(\xi,\bar{\xi})$ without the addition of counterterms) if the hamiltonian operator $\mathbb{H}(a,a^\dagger)$ is written in (anti-)symmetric form: for the present simple model it simply means $\mathbb{H}_S(a,a^\dagger)=c_0+c_1a+c_2a^\dagger+c_3(aa^\dagger-a^\dagger a)$. In general the hamiltonian will not have such form and it is necessary to order it $\mathbb{H}=\mathbb{H}_S+$ "counterterms", where "counterterms" come from anticommuting a and $a^\dagger$ in order to put $\mathbb{H}$ in symmetrized form. The present ordering is called Weyl-ordering. For details about Weyl ordering in bosonic and fermionic path integrals see [4].

Let us now compute the trace of the evolution operator. It yields

$$\text{tr}\ e^{-it\mathbb{H}}=\int d\eta d\bar{\lambda}\ e^{\bar{\lambda}\eta}\langle\bar{\lambda}|e^{-it\mathbb{H}}|\eta\rangle=\int d\eta\int_{\xi(0)=\eta}^{\bar{\xi}(1)=\bar{\lambda}}D\bar{\xi}D\xi d\bar{\lambda}\ e^{\bar{\lambda}(\eta+\xi(1))}e^{i\int_0^1(i\bar{\xi}\dot{\xi}-H)} \tag{3.16}$$

then the integral over $\bar{\lambda}$ give a Dirac delta that can be integrated with respect to η. Hence,

$$\text{tr } e^{-it\mathbb{H}} = \int_{\xi(0)=-\xi(1)} D\bar{\xi} D\xi \; e^{i\int_0^1 (i\bar{\xi}\dot{\xi} - H(\xi,\bar{\xi}))} \tag{3.17}$$

where we notice that the trace in the fermionic variables corresponds to a path integral with anti-periodic boundary conditions (ABC), as opposed to the periodic boundary conditions of the bosonic case. Finally, we can rewrite the latter by using real (Majorana) fermions defined as

$$\xi = \frac{1}{\sqrt{2}}\left(\psi^1 + i\psi^2\right), \quad \bar{\xi} = \frac{1}{\sqrt{2}}\left(\psi^1 - i\psi^2\right) \tag{3.18}$$

and

$$\text{tr } e^{-it\mathbb{H}} = \int_{ABC} D\psi \; e^{i\int_0^1 (\frac{i}{2}\psi_a \dot{\psi}^a - H(\psi))} \; . \tag{3.19}$$

In particular

$$2 = \text{tr}\mathbb{1} = \int_{ABC} D\psi \; e^{i\int_0^1 \frac{i}{2}\psi_a \dot{\psi}^a} \, , \quad a = 1, 2 \; . \tag{3.20}$$

For an arbitrary number of fermionic operator pairs $a_i \, , \; a_i^\dagger \, , \;\; i = 1, ..., l$, the latter of course generalizes to

$$2^{D/2} = \text{tr}\mathbb{1} = \int_{ABC} D\psi \; e^{i\int_0^1 \frac{i}{2}\psi_a \dot{\psi}^a} \, , \quad a = 1, ..., D = 2l \tag{3.21}$$

that sets the normalization of the fermionic path integral with anti-periodic boundary conditions. The latter fermionic action plays a fundamental role in the description of relativistic spinning particles, that is the subject of the next section.

3.1 Problems for Section 3

(1) Show that the ket and bra defined in (3.3) are eigenstates of a and $a^\dagger$ respectively.

(2) Demonstrate properties (3.4)-(3.7).

(3) Test property (3.8) using $A = \mathbb{1}$.

(4) Obtain the equations of motion from action (3.15) and check that the boundary terms cancel.

4 Relativistic particles: bosonic particles and $O(N)$ spinning particles

We consider a generalization of the previous results to relativistic particles in flat space. In order to do that we start analyzing particle models at the classical level, then consider their quantization, in terms of canonical quantization and path integrals.

4.1 Bosonic particles: symmetries and quantization. The worldline formalism

For a nonrelativistic free particle in d-dimensional space, at the classical level we have

$$S[\mathbf{x}] = \frac{m}{2}\int_0^t d\tau\ \dot{\mathbf{x}}^2\,, \quad \mathbf{x} = (x^i)\,,\ i = 1, ..., d \tag{4.1}$$

that is invariant under a set of continuous global symmetries that correspond to an equal set of conserved charges.

- time translation $\delta x^i = \xi \dot{x}^i \ \longrightarrow\ E = \frac{m}{2}\dot{\mathbf{x}}^2$, the energy
- space translations $\delta x^i = a^i \ \longrightarrow\ P^i = m\dot{x}^i$, linear momentum
- spatial rotations $\delta x^i = \theta^{ij}x^j \ \longrightarrow\ L^{ij} = m(x^i\dot{x}^j - x^j\dot{x}^i)$, angular momentum
- Galilean boosts $\delta x^i = v^i t \ \longrightarrow\ x_0^i :\ x^i = x_0^i + P^i t$, center of mass motion.

These symmetries are isometries of a one-dimensional euclidean space (the time) and a three-dimensional euclidean space (the space). However the latter action is not, of course, Lorentz invariant.

A Lorentz-invariant generalization of the free-particle action can be simply obtained by starting from the Minkowski line element $ds^2 = -dt^2 + d\mathbf{x}^2$. For a particle described by $\mathbf{x}(t)$ we have $ds^2 = -(1 - \dot{\mathbf{x}}^2)dt^2$ that is Lorentz-invariant and measures the (squared) proper time of the particle along its path. Hence the action (referred to as geometric action) for the massive free particle reads

$$S[\mathbf{x}] = -m\int_0^t dt\sqrt{1 - \dot{\mathbf{x}}^2} \tag{4.2}$$

that is, by construction, invariant under the Poincarè group of transformations

- $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu\,, \quad x^\mu = (t, x^i)\,,\ \Lambda \in SO(1,3)\,,\ a^\mu \in \mathbb{R}^{1,3}$

the isometry group of Minkowski space. Conserved charges are four-momentum, angular momentum and center of mass motion (from Lorentz boosts). The above action can be reformulated by making x^0 a dynamical field as well in order to render the action explicitly Lorentz-invariant. It can be achieved by introducing a gauge symmetry. Hence

$$S[x] = -m\int_0^1 d\tau\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \tag{4.3}$$

where now $\dot{} = \frac{d}{d\tau}$, and $\eta_{\mu\nu}$ is the Minkowski metric. The latter is indeed explicitly Lorentz-invariant as it is written in four-tensor notation and it also gauge invariant upon the reparametrization $\tau \to \tau'(\tau)$. Action (4.2) can be recovered upon gauge choice $x^0 = t\tau$. Yet another action for the relativistic particle can be obtained by introducing a gauge fields, the einbein e, that renders explicit the above gauge invariance.

$$S[x, e] = \int_0^1 d\tau\left(\frac{1}{2e}\dot{x}^2 - \frac{m^2 e}{2}\right)\ . \tag{4.4}$$

For an infinitesimal time reparametrization

$$\delta\tau = -\xi(\tau)\,, \quad \delta x^\mu = \xi\dot{x}^\mu\,, \quad \delta e = \big(e\xi\big)^\bullet \tag{4.5}$$

we have $\delta S[x,e] = \int d\tau\big(\xi L\big)^\bullet = 0$. Now a few comments are in order: (a) action (4.3) can be recovered by replacing e with its on-shell expression; namely,

$$0 = m^2 + \frac{1}{e^2}\dot{x}^2 \quad \Rightarrow \quad e = \frac{\sqrt{-\dot{x}^2}}{m}\ ; \tag{4.6}$$

(b) unlike the above geometric actions, expression (4.4), that is known as Brink-di Vecchia-Howe action, is also suitable for massless particles; (c) equation(4.4) is quadratic in x and therefore is more easily quantizable. In fact we can switch to phase-space action by taking $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \dot{x}_\mu/e$ (e has vanishing conjugate momentum, it yields a constraint)

$$S[x,p,e] = \int_0^1 d\tau\Big[p_\mu\dot{x}^\mu - e\frac{1}{2}\big(p^2+m^2\big)\Big] \tag{4.7}$$

which is like a standard (nonrelativistic) hamiltonian action, with hamiltonian $\mathcal{H} = e\frac{1}{2}\big(p^2+m^2\big) \equiv eH_0$ and phase space constraint $H_0 = 0$. The constraint H_0 works also as gauge symmetry generator $\delta x^\mu = \{x^\mu, \xi H_0\} = \xi p^\mu$ and, by requiring that $\delta S = 0$, one gets $\delta e = \dot{\xi}$. Here $\{\ ,\ \}$ are Poisson brackets.

Upon canonical quantization the dynamics is governed by a Schrödinger equation with hamiltonian operator $\mathcal{H}$ and the constraint is an operatorial constraint that must be imposed on physical states

$$i\partial_\tau|\phi(\tau)\rangle = \mathcal{H}|\phi(\tau)\rangle = eH_0|\phi(\tau)\rangle = 0 \tag{4.8}$$

$$\Rightarrow \quad (p^2+m^2)|\phi\rangle \tag{4.9}$$

with $|\phi\rangle$ being τ independent. In the coordinate representation (4.9) is nothing but Klein-Gordon equation. In conclusion the canonical quantization of the relativistic, 1d-reparametrization invariant particle action (4.4) yields Klein-Gordon equation for the wave function. This is the essence of the "worldline formalism" that uses (the quantization) of particle models to obtain results in quantum field theory (see [6] for a review of the method). Another important comment here is that the local symmetry (1d reparametrization) ensures the propagation of physical degrees of freedom; i.e. it guarantees unitarity. Before switching to path integrals let us consider the coupling to external fields: in order to achieve that, one needs to covariantize the particle momentum in H_0. For a coupling to a vector field

$$p_\mu \quad \to \quad \pi_\mu = p_\mu - qA_\mu \quad \Rightarrow \quad \{\pi_\mu, \pi_\nu\} = qF_{\mu\nu} \tag{4.10}$$

$$H_0 = \frac{1}{2}\Big(\eta^{\mu\nu}\pi_\mu\pi_\nu + m^2\Big) \tag{4.11}$$

and

$$S[x,p,e;A_\mu] = \int_0^1 d\tau\Big[p_\mu\dot{x}^\mu - e\frac{1}{2}\big(\pi^2+m^2\big)\Big] \tag{4.12}$$

with q being the charge of the particle and $F_{\mu\nu}$ the vector field strength. Above the vector field is in general nonabelian $A_\mu = A^a_\mu T_a$ and $T_a \in$ Lie algebra of a gauge group G. For the bosonic particle, (minimal) coupling to gravity is immediate to achieve, and amounts to the replacement $\eta_{\mu\nu} \to g_{\mu\nu}(x)$; for a spinning particle it would be rather more involved (see e.g. [7]), but will not be treated here. In order to switch to configuration space we just solve for π_μ in (4.12), $\pi_\mu = \eta_{\mu\nu}\dot{x}^\nu/e$ and get

$$S[x, e; A_\mu] = \int_0^1 d\tau \Big[\frac{1}{2e}\dot{x}^2 - \frac{e}{2}m^2 + q\dot{x}^\mu A_\mu\Big] \tag{4.13}$$

so, although the hamiltonian involves a term quadratic in A_μ, the coupling between particle and external vector field in configuration space is linear. Moreover, for an abelian vector field, action (4.13) is gauge invariant upon $A_\mu \to A_\mu + \partial_\mu \alpha$. For a nonabelian vector field, whose gauge transformation is $A_\mu \to U^{-1}(A_\mu - i\partial_\mu)U$, with $U = e^{i\alpha}$ action (4.13) is not gauge invariant; however in the path integral the action enters in the exponent so it is possible to give the following gauge-invariant prescription

$$e^{iS[x,e;A_\mu]} \quad \longrightarrow \quad \mathrm{tr}\left(\mathcal{P}e^{iS[x,e;A_\mu]}\right) \tag{4.14}$$

i.e. we replace the simple exponential with a Wilson line. Here $\mathcal{P}$ defines the "path ordering" that, for the worldline integral $e^{iq\int_0^1 \dot{x}^\mu A_\mu}$, is nothing but the "time-ordering" mentioned in footnote 1; namely

$$\mathcal{P}e^{iq\int_0^1 \dot{x}^\mu A_\mu} = 1 + iq\int_0^1 d\tau\ \dot{x}^\mu A_\mu + (iq)^2 \int_0^1 d\tau_1\ \dot{x}^{\mu_1} A_{\mu_1} \int_0^{\tau_1} d\tau_2\ \dot{x}^{\mu_2} A_{\mu_2} + \cdots \tag{4.15}$$

that transforms covariantly $\mathcal{P}e^{iq\int_0^1 \dot{x}^\mu A_\mu} \to U^{-1}\mathcal{P}e^{iq\int_0^1 \dot{x}^\mu A_\mu}U$, so that the trace is gauge-invariant, and that, for abelian fields, reduces to the conventional expansion for the exponential.

Let us now consider a path integral for the action (4.13). For convenience we consider its Wick rotated ($i\tau \to \tau$) version (we also change $q \to -q$)

$$S[x, e; A_\mu] = \int_0^1 d\tau \Big[\frac{1}{2e}\dot{x}^2 + \frac{e}{2}m^2 + iq\dot{x}^\mu A_\mu\Big] \tag{4.16}$$

for which the path integral formally reads

$$\int \frac{DxDe}{\mathrm{Vol\ (Gauge)}}\ e^{-S[x,e;A_\mu]} \tag{4.17}$$

where "Vol (Gauge)" refers to the fact that we have to divide out all configurations that are equivalent upon gauge symmetry, that in this case reduces to 1d reparametrization. The previous path integral can be taken over two possible topologies: on the *line* where $x(0) = x'$ and $x(1) = x$, and on the *circle* for which bosonic fields have periodic boundary conditions, in particular $x(0) = x(1)$. However, such path integrals can be used to compute more generic tree-level and (multi-)loop graphs [8, 9].

4.1.1 QM Path integral on the line: QFT propagator

Worldline path integrals on the line are linked to quantum field theory propagators. In particular, for the above 1d-reparametrization invariant bosonic model, coupled to external abelian vector field, one obtains the full propagator of scalar quantum electrodynamics (QED), i.e. a scalar propagator with insertion of an arbitrary number of couplings to A_μ.

On the line we keep fixed the extrema of $x(\tau)$ and the gauge parameter is thus constrained as $\xi(0) = \xi(1) = 0$, and the einbein can be gauge-away to an arbitrary positive constant $\hat{e} \equiv 2T$ where

$$2T \equiv \int_0^1 d\tau \; e \,, \quad \delta(2T) = \int_0^1 d\tau \; \big(e\xi\big)^\bullet = 0 \tag{4.18}$$

and therefore

$$De = dT D\xi \tag{4.19}$$

where $D\xi$ is the measure of the gauge group. Moreover there are no Killing vector as $(\hat{e}\xi)^\bullet = 0$ on the line has only a trivial solution $\xi = 0$. Hence the gauge-fixed path integral reads

$$\Big\langle \phi(x)\phi(x') \Big\rangle_A = \int_0^\infty dT \int_{x(0)=x'}^{x(1)=x} Dx \; e^{-S[x,2T;A_\mu]} \tag{4.20}$$

and

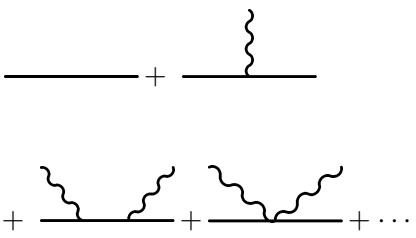
$$S[x,2T;A_\mu] = \int_0^1 d\tau \Big(\frac{1}{4T}\dot{x}^2 + Tm^2 + iq\dot{x}^\mu A_\mu \Big) \tag{4.21}$$

is the gauge-fixed action. For $A_\mu = 0$ it is easy to convince oneself that (4.20) reproduces the free bosonic propagator; in fact

$$\int_0^\infty dT \int_{x(0)=x'}^{x(1)=x} Dx \; e^{-\int_0^1 (\frac{\dot{x}^2}{4T} + Tm^2)} = \int_0^\infty dT \; \langle x| e^{-T(\mathbb{p}^2+m^2)} |x'\rangle = \langle x| \frac{1}{\mathbb{p}^2+m^2} |x'\rangle \; . \tag{4.22}$$

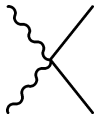
In perturbation theory, about trivial vector field background, with perturbation by $A_\mu(x) = \sum_i \epsilon^i_\mu e^{ip_i\cdot x}$, i.e. sum of external photons, expression (4.20) is nothing but the sum of the following Feynman diagrams

$$\int_0^\infty dT \int_{x(0)=x'}^{x(1)=x} Dx \; e^{-S[x,2T;A_\mu]} =$$



(4.23)

as two types of vertices appear in scalar QED

$$iqA_\mu(\bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi}) \quad \longrightarrow$$  $$, \qquad q^2 A_\mu A^\mu \bar{\phi}\phi \quad \longrightarrow$$  (4.24)

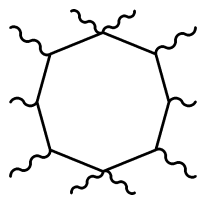
i.e. the linear vertex and the so-called "seagull" vertex. It is interesting to note that the previous expression for the propagator of scalar QED was already proposed by Feynman in his famous "Mathematical formulation of the quantum theory of electromagnetic interaction," [10] where he also included the interaction with an arbitrary number of virtual photons emitted and re-absorbed along the trajectory of the scalar particle.

4.1.2 QM Path integral on the circle: one loop QFT effective action

Worldline path integrals on the circle are linked to quantum field theory one loop effective actions. With the particle model of (4.16) it yields the one loop effective action of scalar QED. The gauge fixing goes similarly to previous case, except that on the circle we have periodic conditions $\xi(0) = \xi(1)$. This leaves a non-trivial solution, $\xi =$ constant, for the Killing equation that corresponds to the arbitrariness on the choice of origin of the circle. One takes care of this further symmetry, dividing by the length of the circle itself. Therefore

$$\Gamma[A_\mu] = \int_0^\infty \frac{dT}{T} \int_{PBC} Dx \; e^{-S[x,2T;A_\mu]} \tag{4.25}$$

yields the worldline representation for the one loop effective action for scalar QED. Perturbatively the latter corresponds to the following sum of one particle irreducible Feynman diagrams

$$\Gamma[A_\mu] = \sum$$  (4.26)

i.e. it corresponds to the sum of one-loop photon amplitudes. The figure above is meant to schematically convey the information that scalar QED effective action involves both types of vertices. Further details about the many applications of the previous effective action representation will be given by Christian Schubert in his lectures set [11].

4.2 Spinning particles: symmetries and quantization. The worldline formalism

We can extend the phase space bosonic form by adding fermionic degrees of freedom. For example we can add Majorana worldline fermions that carry a space-time index μ and an internal index i and get

$$I_{sf}[x,p,\psi] = \int_0^1 d\tau \Big(p_\mu \dot{x}^\mu + \frac{i}{2}\psi_{\mu i}\dot{\psi}_i^\mu \Big) \;, \quad i = 1,\ldots,N \;. \tag{4.27}$$

The latter expression is invariant under the following set of continuous global symmetries, with their associated conserved Noether charges

- time translation: $\delta x^\mu = \xi p^\mu\,,\ \delta p_\mu = \delta\psi_i^\mu = 0 \ \longrightarrow\ H_0 = \frac{1}{2}p_\mu p^\mu$
- supersymmetries: $\delta x^\mu = i\epsilon_i\psi_i^\mu\,,\ \delta p^\mu = 0\,,\ \delta\psi_i^\mu = -\epsilon_i p^\mu \ \longrightarrow\ Q_i = p_\mu\psi_i^\mu$
- $O(N)$ rotations: $\delta x^\mu = \delta p_\mu = 0\,,\ \delta\psi_i^\mu = \alpha_{ij}\psi_j^\mu \ \longrightarrow\ J_{ij} = i\psi_{\mu i}\psi_j^\mu$

with arbitrary constant parameters ξ, ϵ_i, α_{ij}, and $\alpha_{ij} = -\alpha_{ij}$. Conserved charged also work as symmetry generators $\delta z = \{z, g\}$ with $z = (x, p, \psi)$ and $G = \Xi^A G_A \equiv \xi H_0 + i\epsilon_i Q_i + \frac{1}{2}\alpha_{ij}J_{ij}$, and $\{\ ,\ \}$ being graded Poisson brackets; in flat space the generators G_A satisfies a first-class algebra $\{G_A, G_B\} = C_{AB}^C G_C$, see [7] for details. Taking the parameters to be time-dependent we have that the previous symplectic form transforms as

$$\delta I_{sf}[x, p, \psi] = \int_0^1 d\tau\Big(\dot\xi H_0 + i\dot\epsilon_i Q_i + \frac{1}{2}\dot\alpha_{ij}J_{ij}\Big) \tag{4.28}$$

so that we can add gauge fields $E = (e, \chi_i, a_{ij})$ and get the following locally-symmetric particle action

$$S[x, p, \psi, E] = \int_0^1 d\tau\Big(p_\mu\dot x^\mu + \frac{i}{2}\psi_{\mu i}\dot\psi_i^\mu - eH_0 - i\chi_i Q_i - \frac{1}{2}a_{ij}J_{ij}\Big)\ . \tag{4.29}$$

This is a spinning particle model with gauged $O(N)$-extended supersymmetry. The fact that the symmetry algebra is first class ensures that (4.29) is invariant under the local symmetry generated by $G = \Xi^A(\tau)G_A$, provided the fields E transform as

$$\begin{aligned}\delta e &= \dot\xi + 2i\chi_i\epsilon_i\\ \delta\chi_i &= \dot\epsilon_i - a_{ij}\epsilon_j + \alpha_{ij}\chi_j\\ \delta a_{ij} &= \dot\alpha_{ij} + \alpha_{im}a_{mj} - a_{im}\alpha_{mj}\end{aligned} \tag{4.30}$$

from which it is clear that they are gauge fields.

Upon canonical quantization Poisson brackets turn into (anti-)commutators $[p_\mu, x^\nu] = -i\delta_\mu^\nu\,,\ \{\psi_i^\mu, \psi_j^\nu\} = \delta_{ij}\eta^{\mu\nu}$, where $\{\ ,\ \}$ now represent anti-commutators. One possible representation of the previous fermionic algebra, that is nothing but a multi-Clifford algebra, is the spin-basis, where ψ_i^μ are represented as Gamma-matrices. So, in the spin-basis and in bosonic coordinate representation the wave function is a multispinor $\phi_{\alpha_1\cdots\alpha_N}(x)$ where ψ_i^μ acts as Gamma-matrix on the $i-$th $\alpha-$index. First class constraints again act *à la Dirac-Gupta-Bleuler* on the wave function. In particular the susy constraints

$$Q_i|\phi\rangle = 0 \ \longrightarrow\ \big(\gamma^\mu\big)_{\alpha_i\tilde\alpha_i}\partial_\mu\phi_{\alpha_1\cdots\tilde\alpha_i\cdots\alpha_N}(x) = 0 \tag{4.31}$$

amount to N massless Dirac equations, whereas the $O(N)$ constraints

$$J_{ij}|\phi\rangle = 0 \ \longrightarrow\ \big(\gamma^\mu\big)_{\alpha_i\tilde\alpha_i}\big(\gamma_\mu\big)_{\alpha_j\tilde\alpha_j}\phi_{\alpha_1\cdots\tilde\alpha_i\cdots\tilde\alpha_j\cdots\alpha_N}(x) = 0 \tag{4.32}$$

are "irreducibility" constraints, i.e. they impose the propagation of a field that is described by a single Young tableau of $SO(1, D-1)$, with $N/2$ columns and $D/2$ rows. The previous set of constraints yields Bargmann-Wigner equations for spin-$N/2$ fields in flat space. For generic N only particle models in even dimensions are non-empty, whereas for $N \leq 2$

the $O(N)$ constraints are either trivial or abelian and the corresponding spinning particle models can be extended to odd-dimensional spaces.

Coupling to external fields is now much less trivial. Coupling to gravity can be achieved by covariantizing momenta, and thus susy generators; however, for $N > 2$, in a generically curved background the constraints algebra ceases to be first class. For conformally flat spaces the algebra turns into a first-class non-linear algebra that thus describes the propagation of spin-$N/2$ fields in such spaces [7].

4.2.1 $N = 1$ spinning particle: coupling to vector fields

We consider the spinning particle model with $N = 1$ that describes the first quantization of a Dirac field. For the free model, at the classical level, the constraint algebra is simply

$$\{Q, Q\} = -2iH_0\,, \quad \{Q, H_0\} = 0 \tag{4.33}$$

that is indeed first class. To couple the particle model to an external vector field we covariantize the momentum as in (4.10), and consequently

$$Q \equiv \pi_\mu \psi^\mu\,, \quad \{Q, Q\} = -2iH \tag{4.34}$$

with

$$H = \frac{1}{2}\eta^{\mu\nu}\pi_\mu\pi_\nu + \frac{i}{2}qF_{\mu\nu}\psi^\mu\psi^\nu \tag{4.35}$$

and the phase-space locally symmetric action reads

$$S[x, p, \psi, e, \chi; A_\mu] = \int_0^1 d\tau \Big[p_\mu \dot{x}^\mu + \frac{i}{2}\psi_\mu\dot{\psi}^\mu - eH - i\chi Q\Big] \tag{4.36}$$

whereas

$$S[x, \psi, e, \chi; A_\mu] = \int_0^1 d\tau \Big[\frac{1}{2e}\eta_{\mu\nu}(\dot{x}^\mu - i\chi\psi^\mu)(\dot{x}^\nu - i\chi\psi^\nu) + \frac{i}{2}\psi_\mu\dot{\psi}^\mu + q\dot{x}^\mu A_\mu - eq\frac{i}{2}F_{\mu\nu}\psi^\mu\psi^\nu\Big] \tag{4.37}$$

is the locally symmetric configuration space action where, along with the bosonic coupling found previously, a Pauli-type coupling between field strength and spin appears.

4.2.2 QM Path integral on the circle: one loop QFT effective action

We now consider the above spinning particle models on a path integral on the circle, i.e. we consider the one loop effective actions produced by the spin-$N/2$ fields whose first quantization is described by the spinning particle models. On the circle (fermionic) bosonic fields have (anti-)periodic boundary conditions. It is thus not difficult to convince oneself that gravitini χ_i can be gauged-away completely. For the $N = 1$ model of the previous section this yields the spinor QED effective action

$$\Gamma[A_\mu] = \int_0^\infty \frac{dT}{2T}\int_{PBC} Dx \int_{ABC} D\psi\; e^{-S[x,\psi,2T,0;A_\mu]} \tag{4.38}$$

with

$$S[x,\psi,2T,0;A_\mu] = \int_0^1 d\tau \Big[\frac{1}{4T}\dot{x}^2 + \frac{1}{2}\psi_\mu\dot{\psi}^\mu + iq\dot{x}^\mu A_\mu - iTqF_{\mu\nu}\psi^\mu\psi^\nu\Big] \tag{4.39}$$

being the (euclidean) gauge-fixed spinning particle action, that is globally supersymmetric. Perturbatively the previous path integral is the sum of one particle irreducible diagrams with external photons and a Dirac fermion in the loop.

For arbitrary N we will not consider the coupling to external fields as it too much of an involved topic to be covered here. The interested reader may consult the recent manuscript [12] and references therein. Let us consider the circle path integral for the free $O(N)-$extended spinning particle. The euclidean configuration space action can be obtained from (4.29) by solving for the particle momenta and Wick rotating. We thus get

$$S[x,\psi,E] = \int_0^1 d\tau \Big[\frac{1}{2e}\eta_{\mu\nu}(\dot{x}^\mu - \chi_i\psi_i^\mu)(\dot{x}^\nu - \chi_i\psi_i^\nu) + \frac{1}{2}\psi_\mu\dot{\psi}^\mu - \frac{1}{2}a_{ij}\psi_{\mu i}\psi_j^\mu\Big] \tag{4.40}$$

that yields the circle path integral

$$\Gamma = \frac{1}{\text{Vol (Gauge)}}\int_{PBC} DxDeDa \int_{ABC} D\psi D\chi\ e^{-S[x,\psi,E]}\ . \tag{4.41}$$

Using (4.30), with antiperiodic boundary conditions for fermions, gravitini can be gauged away completely, $\chi_i = 0$. On the other hand $O(N)$ gauge fields enter with periodic boundary conditions and cannot be gauged away completely. In fact, as shown in [13] they can be gauged to a skew-diagonal constant matrix parametrized by $n = [N/2]$ angular variables, θ_k. The whole effective action is thus proportional to the number of degrees of freedom of fields described by a Young tableau with n columns and $D/2$ rows. Such Young tableaux correspond to the field strengths of higher-spin fields. For $D = 4$ this involves all possible massless representations of the Poincaré group, that at the level of gauge potentials are given by totally symmetric (spinor-) tensors, whereas for $D > 4$ it corresponds to conformal multiplets only.

4.3 Problems for Section 4

(1) Use the Noether trick to obtain the conserved charges for the free particle described by the geometric action (4.2).

(2) Repeat the previous problem with action (4.3).

(3) Show that the interaction term $L_{int} = q\dot{x}^\mu A_\mu$, with $A^\mu = (\phi, \mathbf{A})$, yields the Lorentz force.

(4) Show that, with time-dependent symmetry parameters, the symplectic form transforms as (4.28).

(5) Show that action (4.39) is invariant under global susy $\delta x^\mu = \epsilon\psi^\mu\ ,\ \delta\psi^\mu = -\frac{1}{2T}\epsilon\dot{x}^\mu$.

5 Functional methods in Quantum Field Theory

We extent the path-integral approach described above to space-time quantum field theory.

5.1 Introduction: correlation functions in the operatorial formalism

The necessity of a field theoretical description is necessary in order to describe physical processes where the nature and/or the number of particles in the final state is different than that of the initial state. We thus quantize a field that represents the creation/absorption of particles. Having in mind collider physics where initial and final asymptotic states are described by free particle states we can parametrize the hamiltonian as $\mathbb{H} = \mathbb{H}_0 + \mathbb{H}_1$ where the first term is the free theory hamiltonian and the second the interaction, and asymptotic states are characterized by momenta and polarizations of incoming (outgoing) particles. Also, we indicate with $|0\rangle$ and $|\Omega\rangle$ the vacuum states of $\mathbb{H}_0$ and $\mathbb{H}$ respectively, with $\mathbb{H}_0|0\rangle = 0$ and $\mathbb{H}|\Omega\rangle = E_0|\Omega\rangle$. As we saw in section 2.3.1 a large-time dynamics only involves transitions from vacuum to vacuum. In particular, one is interested in time-ordered correlations functions of field operators

$$\langle\Omega|\mathcal{T}\big(\phi(x_1)\cdots\phi(x_n)\big)|\Omega\rangle \tag{5.1}$$

as they turn out to be related to elements of the S-matrix, S_{fi} that describes amplitudes of probability between and initial state $|i\rangle$ and a final state $|f\rangle$. Above the operators $\phi(x)$ are field operators in the Heisenberg representation, that evolve in time according to the full hamiltonian. We concentrate on the simplest field theory a scalar field theory for which the field involves scalar particles, i.e. particles with spin zero. In general the above evolution, and the vacuum state $|\Omega\rangle$ are not known and one must rely upon approximate methods, such as perturbation theory. It is thus convenient to use the interaction picture instead, which yields

$$\phi(x) = U^\dagger(t,t_0)\phi_I(x)U(t,t_0) \tag{5.2}$$

where $U(t,t_0) = e^{i\mathbb{H}_0(t-t_0)}e^{-i\mathbb{H}(t-t_0)}$ and $\phi_I(x)$ are respectively the evolution operator and the field operator, in the interaction picture, and t_0 is associated with the Schrodinger operator $\phi_S(\mathbf{x}) = \phi(\mathbf{x},t_0)$, in terms of which

$$\phi(x) = e^{i\mathbb{H}(t-t_0)}\phi(\mathbf{x},t_0)e^{-i\mathbb{H}(t-t_0)} \tag{5.3}$$

$$\phi_I(x) = e^{i\mathbb{H}_0(t-t_0)}\phi(\mathbf{x},t_0)e^{-i\mathbb{H}_0(t-t_0)}\ . \tag{5.4}$$

The evolution operator satisfies the Schrodinger equation

$$i\frac{d}{dt}U(t,t_0) = \mathbb{H}_I(t)U(t,t_0)\,, \quad \mathbb{H}_I(t) = e^{i\mathbb{H}_0(t-t_0)}\mathbb{H}_1 e^{-i\mathbb{H}_0(t-t_0)} \tag{5.5}$$

and can be formally inverted to give

$$U(t,t_0) = \mathcal{T}\exp\Big(-i\int_{t_0}^{t} d\tau\mathbb{H}_I(\tau)\Big)\ . \tag{5.6}$$

The time evolution of the free vacuum state to an infinitely large finite time gives

$$e^{-i\mathbb{H}\Delta t}|0\rangle = \sum_n e^{-iE_n\Delta t}|n\rangle\langle n|0\rangle \underset{t\to\infty}{\longrightarrow} e^{-iE_0\Delta t}|\Omega\rangle\langle\Omega|0\rangle \tag{5.7}$$

with $\Delta t = t + t_0$; it is understood that the limit is taken on a slightly imaginary direction $t \to \infty(1 - i\epsilon)$. From the latter

$$|\Omega\rangle = \lim_{t\to\infty} \frac{e^{-i\mathbb{H}(t+t_0)}|0\rangle}{e^{-iE_0(t+t_0)}\langle\Omega|0\rangle} = \lim_{t\to\infty} \frac{U(t_0,-t)|0\rangle}{e^{-iE_0(t+t_0)}\langle\Omega|0\rangle} \tag{5.8}$$

where in the second passage we used the fact that one can add for free $e^{i\mathbb{H}_0(t+t_0)}$ in front of $|0\rangle$ and that the evolution operator can be written as

$$U(t_2,t_1) = e^{i\mathbb{H}_0(t_2-t_1)}e^{-i\mathbb{H}(t_2-t_1)} = e^{i\mathbb{H}_0(t_2-t')}e^{-i\mathbb{H}(t_2-t_1)}e^{-i\mathbb{H}_0(t_1-t')} \tag{5.9}$$

that implies $U^\dagger(t_2,t_1) = e^{i\mathbb{H}(t_2-t_1)}e^{-i\mathbb{H}_0(t_2-t_1)} = U(t_1,t_2)$. Analogously,

$$\langle\Omega| = \lim_{t\to\infty} \frac{\langle 0|U(t,t_0)}{e^{-iE_0(t-t_0)}\langle 0|\Omega\rangle} \tag{5.10}$$

and, using that $U(t_2,t')U(t',t_1) = U(t_2,t_1)$, one gets

$$1 = \langle\Omega|\Omega\rangle = \lim_{t\to\infty}\langle 0|U(t,-t)|0\rangle\Big(|\langle 0|\Omega\rangle|^2 e^{-i2E_0t}\Big)^{-1} . \tag{5.11}$$

In the very same way one can compute a correlation function of an arbitrary number n of field operators evaluated at different space-time points. For $n = 2$ and $x^0 > y^0$ we have

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \lim_{t\to\infty}\langle 0|U(t,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-t)|0\rangle\Big(|\langle 0|\Omega\rangle|^2 e^{-i2E_0t}\Big)^{-1} \\ &= \lim_{t\to\infty}\frac{\langle 0|U(t,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-t)|0\rangle}{\langle 0|U(t,-t)|0\rangle}\end{aligned} \tag{5.12}$$

and similarly for $x^0 < y^0$ with the position of the two field operators inverted. For arbitrary n, one can thus write the compact formula

$$\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(x_k)|\Omega\rangle = \frac{\langle 0|\mathcal{T}\phi_I(x_1)\cdots\phi_I(x_k)e^{-i\int_{-\infty}^{\infty}d\tau\mathbb{H}_I(\tau)}|0\rangle}{\langle 0|e^{-i\int_{-\infty}^{\infty}d\tau\mathbb{H}_I(\tau)}|0\rangle} \tag{5.13}$$

that allows to obtain (full) vacuum correlators of Heisenberg fields via free-vacuum correlators of fields in the interaction picture, that evolve in time with the free hamiltonian, inserting the time-ordered exponential of the interacting hamiltonian $H_I(t)$. In the Schrodinger picture $\mathbb{H}_1 = \int d^3x\mathcal{H}_1(\phi(\mathbf{x},t_0))$ so that in the interaction picture one simply gets $\mathbb{H}_I(t) = \int d^3x\mathcal{H}_1(\phi_I(x))$. The full correlation function in general cannot be obtained analytically. However for scattering processes with asymptotic states given by free particles one may rely on perturbation theory, expanding the exponent. One thus needs to compute free-vacuum correlators of integer powers of the field operator in the interaction picture,

i.e. $\langle 0|\mathcal{T}\phi_I(x_1)\cdots\phi_I(x_n)|0\rangle$. The two point function, $n=2$, is special and is named (free) "propagator"

$$\langle 0|\mathcal{T}\phi_I(x_1)\phi_I(x_2)|0\rangle \equiv \underbrace{\phi_I(x_1)\phi_I}(x_2)\ . \tag{5.14}$$

It allows to compute the free correlation function of an arbitrary number of fields leading to the so-called Wick's theorem

$$\langle 0|\mathcal{T}\phi_I(x_1)\cdots\phi_I(x_n)|0\rangle = \begin{cases} 0, & n \text{ odd} \\ \sum \text{products of pair} - \text{contracted terms}, & n \text{ even} \end{cases} \tag{5.15}$$

For example, for $n=4$, one gets

$$\begin{aligned}\langle 0|\mathcal{T}\phi_I(x_1)\cdots\phi_I(x_4)|0\rangle &= \underbrace{\phi_I(x_1)\phi_I}(x_2)\,\underbrace{\phi_I(x_3)\phi_I}(x_4) \\ &+ \underbrace{\phi_I(x_1)\phi_I}(x_3)\,\underbrace{\phi_I(x_2)\phi_I}(x_4) + \underbrace{\phi_I(x_1)\phi_I}(x_4)\,\underbrace{\phi_I(x_2)\phi_I}(x_3)\end{aligned} \tag{5.16}$$

and similarly for higher even n. So far in the present section we used operatorial formalism: such formalism however becomes cumbersome to use for more sophisticated theories such as gauge theories. Therefore in the following we introduce the functional, path integral, method to represent and compute correlation functions. Although we derive it for the same scalar field theory as above, the method is also particularly suitable for gauge theories.

5.2 Path integral approach for a relativistic scalar field theory

We consider a scalar field theory whose quantum hamiltonian is given by

$$\mathbb{H} = \int d^3x \left[\frac{1}{2}\Pi^2(x) + \frac{1}{2}\big(\nabla\Phi\big)^2(x) + V(\phi)\right] \tag{5.17}$$

where ϕ here can either be a Schrodinger operator $\phi_S(\mathbf{x})$ or its Heisenberg counterpart $\phi(x) = e^{i\mathbb{H}t}\phi_S(\mathbf{x})e^{-i\mathbb{H}t}$. We can thus imagine to discretize the three-dimensional space and associate a field and its conjugate momentum to a unit cell located at each discrete point. Fields and momenta satisfy canonical commutation relations $[\phi_S(\mathbf{x}), \Pi_S(\mathbf{x}')] = i\delta_{\mathbf{x},\mathbf{x}'}$, so that we can define field eigenstates such that, $\phi_S(\mathbf{x})|\phi_u\rangle = \phi_u(\mathbf{x})|\phi_u\rangle$ where $\phi_u(\mathbf{x})$ is the eigenvalue function associate to the eigenstate $|\phi_u\rangle$. Using the results for particle path integral we can write down the configuration space path integral between the initial state $|\phi_a\rangle$ and the final state $|\phi_b\rangle$

$$\langle\phi_b|e^{-i\mathbb{H}(t-(-t))}|\phi_a\rangle \equiv \langle\phi_b(t)|\phi_a(-t)\rangle = \int_{\{\phi(\mathbf{x},-t)=\phi_a(\mathbf{x})\}}^{\{\phi(\mathbf{x},t)=\phi_b(\mathbf{x})\}} D\phi\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}} \tag{5.18}$$

where $|\phi_u(t)\rangle$ is the eigenstate of the Heisenberg operator $\phi(\mathbf{x},t)$ with eigenvalue $\phi_u(\mathbf{x})$. Inserting the spectral decomposition of unity in terms of eigenstates of the full hamiltonian one gets

$$\int_{\{\phi(\mathbf{x},-t)=\phi_a(\mathbf{x})\}}^{\{\phi(\mathbf{x},t)=\phi_b(\mathbf{x})\}} D\phi\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}} = \sum_n \langle\phi_b|n\rangle\langle n|\phi_a\rangle e^{-i2E_nt} \underset{t\to\infty}{\longrightarrow} \langle\phi_b|\Omega\rangle\langle\Omega|\phi_a\rangle e^{-i2E_0t}\ . \tag{5.19}$$

We can now compute correlation functions. Let us do it for $n = 2$ and let us take for definiteness $x_2^0 > x_1^0$; we can write

$$\int_{\{\phi(\mathbf{x},-t)=\phi_a(\mathbf{x})\}}^{\{\phi(\mathbf{x},t)=\phi_b(\mathbf{x})\}} D\phi\ \phi(x_2)\phi(x_1)\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}}$$
$$= \int D\phi_1(\mathbf{x}) \int D\phi_2(\mathbf{x})\ \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) \int_{\left\{\substack{\phi(\mathbf{x},\,x_1^0)\,=\,\phi_1(\mathbf{x})\\ \phi(\mathbf{x},\,-t)\,=\,\phi_a(\mathbf{x})}\right\}}^{\left\{\substack{\phi(\mathbf{x},\,t)\,=\,\phi_b(\mathbf{x})\\ \phi(\mathbf{x},\,x_2^0)\,=\,\phi_2(\mathbf{x})}\right\}} D\phi\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}} \tag{5.20}$$

where in the second line we fix the values of the field at x_1^0 and x_2^0 and integrate over all possible values. This allows to take out of the path integral the fields $\phi(x_1)$ and $\phi(x_2)$. Now the path integral with four boundary conditions is nothing but the product of three distinct path integrals, from $-t$ to x_1^0, from x_1^0 to x_2^0 and from x_2^0 to t

$$\int D\phi_1(\mathbf{x}) \int D\phi_2(\mathbf{x})\ \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) \int_{\{\phi(\mathbf{x},x_2^0)=\phi_2(\mathbf{x})\}}^{\{\phi(\mathbf{x},t)=\phi_b(\mathbf{x})\}} D\phi\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}}$$
$$\times \int_{\{\phi(\mathbf{x},x_1^0)=\phi_1(\mathbf{x})\}}^{\{\phi(\mathbf{x},x_2^0)=\phi_2(\mathbf{x})\}} D\phi\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}} \int_{\{\phi(\mathbf{x},-t)=\phi_a(\mathbf{x})\}}^{\{\phi(\mathbf{x},x_1^0)=\phi_1(\mathbf{x})\}} D\phi\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}} \tag{5.21}$$

that with the help of (5.18) can be reduced to

$$\int D\phi_1(\mathbf{x}) \int D\phi_2(\mathbf{x})\ \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\langle\phi_b|e^{-i\mathbb{H}(t-x_2^0)}|\phi_2\rangle\langle\phi_2|e^{-i\mathbb{H}(x_2^0-x_1^0)}|\phi_1\rangle\langle\phi_1|e^{-i\mathbb{H}(x_1^0-(-t))}|\phi_a\rangle \tag{5.22}$$

One can now trade the eigenvalues $\phi_u(\mathbf{x}_u)$ with operators $\phi_S(\mathbf{x}_u)$ placed next to the states $|\phi_u\rangle$. It is thus now possible to integrate away the spectral decomposition of unity $\int D\phi_u(\mathbf{x})|\phi_u\rangle\langle\phi_u| = \mathbb{1}$. We are left with

$$\int_{\{\phi(\mathbf{x},-t)=\phi_a(\mathbf{x})\}}^{\{\phi(\mathbf{x},t)=\phi_b(\mathbf{x})\}} D\phi\ \phi(x_2)\phi(x_1)\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}}$$
$$= \langle\phi_b|e^{-i\mathbb{H}t}\ e^{i\mathbb{H}x_2^0}\phi_S(\mathbf{x}_2)e^{-i\mathbb{H}x_2^0}\ e^{i\mathbb{H}x_1^0}\phi_S(\mathbf{x}_1)e^{-i\mathbb{H}x_1^0}\ e^{i\mathbb{H}(-t)}|\phi_a\rangle$$
$$= \langle\phi_b|e^{-i\mathbb{H}t}\ \phi(x_2)\phi(x_1)\ e^{i\mathbb{H}(-t)}|\phi_a\rangle$$
$$= \sum_{n,n'} e^{-i(E_n+E_{n'})t}\langle\phi_b|n\rangle\langle n'|\phi_a\rangle\langle n|\phi(x_2)\phi(x_1)|n'\rangle\ . \tag{5.23}$$

For $x_1^0 > x_2^0$ the result is the same with the position of the two Heisenberg operators exchanged. Of course under the path integral the position is arbitrary as they are (bosonic) functions. Hence, in general

$$\int_{\{\phi(\mathbf{x},-t)=\phi_a(\mathbf{x})\}}^{\{\phi(\mathbf{x},t)=\phi_b(\mathbf{x})\}} D\phi\ \phi(x_1)\cdots\phi(x_k)\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}}$$
$$= \sum_{n,n'} e^{-i(E_n+E_{n'})t}\langle\phi_b|n\rangle\langle n'|\phi_a\rangle\langle n|\mathcal{T}\phi(x_1)\cdots\phi(x_k)|n'\rangle \tag{5.24}$$

and similarly for an arbitrary number of fields. We can finally write a normalized finite-time correlation function as

$$\frac{\int_{\{\phi(\mathbf{x},-t)=\phi_a(\mathbf{x})\}}^{\{\phi(\mathbf{x},t)=\phi_b(\mathbf{x})\}} D\phi\ \phi(x_2)\phi(x_1)\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}}}{\int_{\{\phi(\mathbf{x},-t)=\phi_a(\mathbf{x})\}}^{\{\phi(\mathbf{x},t)=\phi_b(\mathbf{x})\}} D\phi\ e^{i\int_{-t}^{t}dt\int d^3x\mathcal{L}}} = \frac{\sum_{n,n'} e^{-i(E_n+E_{n'})t}\langle\phi_b|n\rangle\langle n'|\phi_a\rangle\langle n|\mathcal{T}\phi(x_2)\phi(x_1)|n'\rangle}{\sum_n e^{-i2E_nt}\langle\phi_b|n\rangle\langle n|\phi_a\rangle} \tag{5.25}$$

that, with an imaginary time gives the finite-temperature correlation function. At zero temperature (infinite time) only the vacuum survives and a drastic simplification occurs: all the details depending on the boundary states $|\phi_u\rangle$ drop out. In other words the zero-temperature correlation function does not depend upon the boundary conditions: we can thus neglect to write them down. Therefore,

$$\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(x_k)|\Omega\rangle = \frac{\int D\phi\ \phi(x_1)\cdots\phi(x_k)\ e^{i\int d^4x\mathcal{L}}}{\int D\phi\ e^{i\int d^4x\mathcal{L}}} \tag{5.26}$$

that turns out to be quite similar in form to its operatorial counterpart (5.13); here $\int d^4x = \int_{-\infty}^{\infty} dt\int d^3x$. In particular we can identify

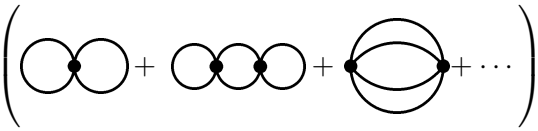
$$\begin{aligned}\int D\phi\ e^{i\int d^4x\mathcal{L}} = \mathcal{N}\langle 0|e^{-i\int_{-\infty}^{\infty}d\tau\mathbb{H}_I(\tau)}|0\rangle &= \mathcal{N}\sum \text{vacuum diagrams}\\ &= \mathcal{N}\exp\Big(\sum \text{connected vacuum diagrams}\Big)\end{aligned} \tag{5.27}$$

with the normalization factor $\mathcal{N}$ being the path integral for the free theory,

$$\mathcal{L}_f = -\frac{1}{2}\Big[\big(\partial\phi\big)^2 + m^2\phi^2\Big] \tag{5.28}$$

$$\mathcal{N} = \int D\phi\ e^{i\int d^4x\mathcal{L}_f} \tag{5.29}$$

Let us for example consider a self-interacting scalar with quartic interacting lagrangian $\mathcal{L}_{int} = \lambda\phi^4$. The perturbative expansion for such a theory, up to three loops, schematically reads

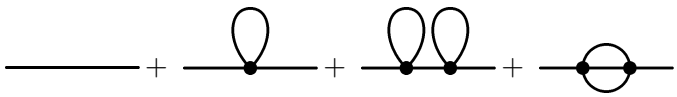
$$\int D\phi\ e^{i\int d^4x\mathcal{L}} = \mathcal{N}\exp\Bigg($$  $$\Bigg) \tag{5.30}$$

where lines represent massive scalar propagators from the free theory (5.28) and quartic vertices come from perturbative insertions of the interaction, from the Taylor expansion of $e^{i\int\mathcal{L}_{int}}$. In the full k-point correlation function (cfr. eq. (5.26)) the only effect of the quotient is to remove the above vacuum diagrams and it thus allows to compactly rewrite the correlation function as

$$\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(x_k)|\Omega\rangle = \int_{\text{v.d.e.}} D\phi\ \phi(x_1)\cdots\phi(x_k)\ e^{i\int d^4x\mathcal{L}} \tag{5.31}$$

with v.d.e. standing for "vacuum diagrams excluded". Diagrammatically, for $k = 2$, at the two-loop level, we have

$$\langle\Omega|\mathcal{T}\phi(x_1)\phi(x_2)|\Omega\rangle =$$



$$+ \cdots \tag{5.32}$$

Below we define a set of generating functionals that allow to compute such correlation functions.

5.2.1 Generating functionals for a massive scalar field theory

In order to compute a generic correlation function

$$G_0^{(n)}(x_1, \ldots, x_n) = \langle 0|\mathcal{T}\phi(x_1)\cdots\phi(x_n)|0\rangle \tag{5.33}$$

in the free massive theory (5.28) we define the functional

$$Z_f[J] \equiv \int D\phi \; e^{iS_f[\phi]+iJ\cdot\phi} \tag{5.34}$$

where "$\cdot$" now indicates the scalar product defined by the four-dimensional integration. We thus have

$$G_f^{(n)}(x_1, \ldots, x_n) = \frac{(-i)^n}{Z_f[0]} \frac{\delta^n Z_f[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0} \; . \tag{5.35}$$

The free action can be rewritten as

$$S_f = -\frac{1}{2}\int d^4x\phi(-\Box + m^2)\phi \equiv -\frac{1}{2}\phi\cdot D\cdot\phi \; , \tag{5.36}$$

using partial integration, and defining the kinetic operator $D(x, x') = \delta^{(4)}(x-x')(-\Box+m^2)$. The functional (5.34) is thus nothing but a Gaussian integral with a linear source term. Then, upon, completing the square one can easily compute the generator

$$Z_f[J] = Z_f[0] \; e^{\frac{i}{2}J\cdot D^{-1}\cdot J} \tag{5.37}$$

so that $G_f^{(2k+1)} = 0$ and

$$G_f^{(2)}(x, x') = -iD^{-1}(x, x') =$$

(5.38)

is the free propagator. It satisfies

$$\mathbb{1} = iD\cdot(-i)D^{-1} \quad \text{i.e.} \quad i(-\Box + m^2)G_f^{(2)}(x - x') = \delta^{(4)}(x - x') \tag{5.39}$$

that in Fourier space takes the common form

$$\widetilde{G}_f^{(2)}(p) = \frac{-i}{p^2 + m^2} \ . \tag{5.40}$$

Since the functional generator in the free theory is Gaussian, the correlation function of an arbitrary even number of fields is a product of two-point functions, that is once again a reformulation of the Wick's theorem seen before. In particular, for $n = 4$, we have

$$\begin{aligned} G_f^{(4)}(x_1, x_2, x_3, x_4) &= G_f^{(2)}(x_1, x_2) G_f^{(2)}(x_3, x_4) + G_f^{(2)}(x_1, x_3) G_f^{(2)}(x_2, x_4) \\ &\quad + G_f^{(2)}(x_1, x_4) G_f^{(2)}(x_2, x_3) \qquad (5.41) \\ &= \qquad + \qquad + \qquad (5.42) \end{aligned}$$

and similarly for higher n. In other words in absence of interaction vertices only the two-point function is a "connected" correlation function: without vertices only two points can be connected together (with a propagator). However also in generically interacting theories it is crucial to be able to separate a generic correlation function from a connected one as only the latter are important in order to compute for S-matrix elements; in presence of interaction vertices connected correlation functions are those function that are represented by geometrically connected diagrams, i.e. diagrams where all external points are connected together with propagators and vertices. Therefore it is important to have a functional, $W[J]$, that only yields connected correlation functions; it is defined as

$$\frac{Z[J]}{Z[0]} \equiv e^{iW[J]} \tag{5.43}$$

and connected correlation functions are defined as

$$W^{(n)}(x_1, \ldots, x_n) = (-i)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1) \ldots \delta J(x_n)} \bigg|_{J=0} \ . \tag{5.44}$$

For the free theory, using (5.37), the above functional reduces to

$$W_f[J] = \frac{i}{2} J \cdot G_f^{(2)} \cdot J = \frac{1}{2} J \cdot D^{-1} \cdot J \tag{5.45}$$

and

$$W_f^{(2)}(x_1, x_2) = G_f^{(2)}(x_1, x_2) \tag{5.46}$$

$$W_f^{(n)}(x_1, \ldots, x_n) = 0 \, , \quad n \neq 2 \ . \tag{5.47}$$

Again with the intention of generalization to an interacting theory we define a third generating functional named "effective action" or generating functional of one-particle irreducible (1PI) correlation functions, that is correlation functions whose Feynman diagram cannot but separated in two parts by cutting a single propagator. It is defined by means of a Legendre transformation of $W_f[J]$; in strict analogy with analytical mechanics $W[J, A]$ plays

the role of the lagrangian and (apart from a sign) $\Gamma[\varphi, A]$ the role of the hamiltonian (A here being a possible external field), with

$$\Gamma[\varphi, A] \equiv W[J, A] - J \cdot \varphi \tag{5.48}$$

$$\varphi(x) \equiv \frac{\delta W[J]}{\delta J(x)} \quad \longrightarrow \quad J(x) = J[\varphi]|_x \tag{5.49}$$

so that the "classical field" φ plays the role of canonical momentum, J plays that of derivative and the external field A that of canonical coordinate. Notice that the inversion passage indicated in (5.49), necessary in order to have Γ only dependent on φ and A, is possible only if there are no constraints. When (as in the above free theory) this is the case one can write (let us take $A = 0$ for simplicity)

$$J(x) = -\frac{\delta \Gamma[\varphi]}{\delta \varphi} \ . \tag{5.50}$$

For the free theory the effective action is exactly solvable: we have

$$\varphi = D^{-1} \cdot J \quad \longrightarrow \quad J = D \cdot \varphi \tag{5.51}$$

$$\Gamma_f[\varphi] = W_f[J] - J \cdot \varphi = -\frac{1}{2} \varphi \cdot D \cdot \varphi = S_f[\varphi] \tag{5.52}$$

and the effective action (in this free case) coincides with the classical action. One-particle irreducible correlation functions are defined as

$$\Gamma^{(n)}(x_1, \ldots, x_n) = (-i)^{n-1} \frac{\delta^n \Gamma[\varphi]}{\delta\varphi(x_1) \cdots \delta\varphi(x_n)} \bigg|_{\varphi = \langle\phi\rangle} \tag{5.53}$$

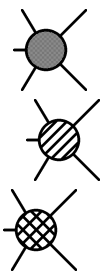
that for the free theory case simply reduce to

$$\Gamma_f^{(2)}(x, x') = iD(x, x') \tag{5.54}$$

$$\Gamma_f^{(n)}(x_1, \ldots, x_n) = 0 \, , \quad n \neq 0 \tag{5.55}$$

in this case $\langle\phi\rangle = W_f^{(1)} = 0$. Now we would like to generalize the relations between different correlation functions for a generically interacting (scalar) field theory. To achieve this task we find it useful to use the shortcuts

$$\begin{aligned}
&J(x_i) \longrightarrow J^i \, , \quad \frac{\delta}{\delta J(x_i)} \longrightarrow \partial_i \\
&\varphi(x_i) \longrightarrow \varphi_i \, , \quad \frac{\delta}{\delta \varphi(x_i)} \longrightarrow \partial^i \\
&G^{(n)}(x_{i_1}, \ldots, x_{i_n}) \longrightarrow G_{i_1 \ldots i_n} = \\
&W^{(n)}(x_{i_1}, \ldots, x_{i_n}) \longrightarrow W_{i_1 \ldots i_n} = \\
&\Gamma^{(n)}(x_{i_1}, \ldots, x_{i_n}) \longrightarrow \Gamma^{i_1 \ldots i_n} =
\end{aligned} \tag{5.56}$$



For the generating functional

$$Z[J] \equiv \int D\phi \; e^{iS_f[\phi]+iS_{int}[\phi]+iJ\cdot\phi} \; , \tag{5.57}$$

treating the interaction perturbatively (i.e. expanding the exponent), we can trade ϕ for $-i\frac{\delta}{\delta J}$ in the interaction and can thus formally write

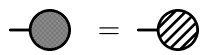
$$Z[J] = e^{iS_{int}[-i\delta/\delta J]} Z_f[J] \tag{5.58}$$

that yields the perturbative (loop) expansion for functional in terms of the vertices included in S_{int} and free propagator from Z_f. The definition(5.43) allows to write full correlation functions $G^{(n)}$ in terms of connected correlation functions $W^{(n)}$ in tun yielding a proof that W only includes connected correlation functions. Let us see that for $n \leq 3$. From

$$\partial_i W[J] = -i \frac{1}{Z[J]} \partial_i Z[J] \; , \tag{5.59}$$

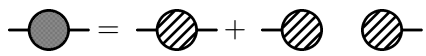
for the one-point function we simply have

$$W_i = W^{(1)}(x_i) = G_i = G^{(1)}(x_i) \, , \tag{5.60}$$



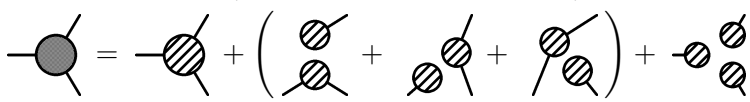
The two-point function can be simply obtained by deriving once again expression (5.59),

$$G_{ij} = W_{ij} + W_i W_j \, , \tag{5.61}$$



from which it is already clear that W represents connected functions. The three-point function can be obtained by a further derivation of (5.59), that yields

$$G_{ijk} = W_{ijk} + \Big(W_{ij}W_k + W_{jk}W_i + W_{ik}W_j \Big) + W_i W_j W_k \tag{5.62}$$



and similarly for higher order functions. We switch now to the relation between connected correlation functions and one-particle irreducible one (also called proper vertices). Let us first stress that the classical field defined in (5.49) is a functional of J, i.e. it really is $\varphi[J]|_x$, in particular

$$\begin{aligned} &\varphi[0]|_x = W^{(1)}(x) = \langle \Omega | \phi(x) | \Omega \rangle \\ &\Rightarrow \; \varphi(x) = \langle \Omega | \phi(x) | \Omega \rangle_J \end{aligned} \tag{5.64}$$

in other words the classical field is the one-point correlation function with open source; we thus define the deviation $\eta(x) = \varphi(x) - \varphi[0]|_x$, that vanishes at $J = 0$.

With the shortcut notation defined above we have that $\varphi_i = \partial_i W$ and $J^i = -\partial^i \Gamma$ and the 1PI two-point function reads

$$\Gamma^{ij} = -i\partial^i \partial^j \Gamma|_{\eta=0} = i\partial^i J^j|_{J=0} = (W^{-1})^{ij} \; . \tag{5.65}$$

The last equality here can be easily proved using the chain rule as follows

$$\delta^i_j = \partial_j J^i = \partial_j \varphi_k \; \partial^k J^i = \partial_j \partial_k W \; \partial^k J^i = -\partial_j \partial_k W \; \partial^k \partial^i \Gamma \tag{5.66}$$

where

$$\partial_j \varphi_k = \partial_j \partial_k W \,, \quad \partial^k J^i = -\partial^k \partial^i \Gamma \tag{5.67}$$

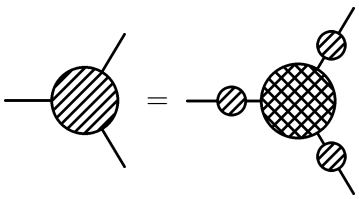
and setting $J = \eta = 0$. From the free theory it is easy to see what equation (5.65) represents: Γ^{ij} is the kinetic operator and W_{ij} its inverse, the propagator. Again using the chain rule and the previous expressions is straightforward to obtain relations for n-point functions. For $n = 3$, we differentiate (5.66) and get

$$\begin{aligned} 0 = -\partial_k \delta^i_j &= \partial_k \Big(\partial_j \partial_l W \; \partial^l \partial^i \Gamma \Big) = \partial_k \partial_j \partial_l W \; \partial^l \partial^i \Gamma + \partial_j \partial_l W \; \partial_k \partial^l \partial^i \Gamma \\ &= \partial_k \partial_j \partial_l W \; \partial^l \partial^i \Gamma + \partial_j \partial_l W \partial_k \partial_{k'} W \; \partial^{k'} \partial^l \partial^i \Gamma \end{aligned} \tag{5.68}$$

where in the last line we used expression (5.67). Setting the sources to zero leads to

$$W_{ijk} = W_{ii'} W_{jj'} W_{kk'} \; \Gamma^{i'j'k'} \tag{5.69}$$

that can be diagrammatically represented as



(5.70)

In the expanded notation the latter reads

$$W^{(3)}(x,y,z) = \int d^4x' d^4y' d^4z' \; W^{(2)}(x,x') W^{(2)}(y,y') W^{(2)}(z,z') \; \Gamma^{(3)}(x',y',z') \; . \tag{5.71}$$

In Figure 3 is depicted a quite non-trivial connected three-point function from a scalar theory with cubic and quartic interaction $g\phi^3 + \lambda\phi^4$, that however exemplifies how the previous decomposition works: the dashed lines are shown to indicate the separation between connected the two-point functions and the 1PI three-point function.

For $n = 4$, one obtains

$$\begin{aligned} W_{ijkl} &= W_{ii'} W_{jj'} W_{kk'} W_{ll'} \; \Gamma^{i'j'k'l'} \\ &\quad + \Big(W_{lii'} W_{jj'} W_{kk'} + W_{ii'} W_{ljj'} W_{kk'} + W_{ii'} W_{jj'} W_{lkk'} \Big) \Gamma^{i'j'k'} \end{aligned} \tag{5.72}$$

whose diagrammatic representation is left as an exercise for the reader (problem n. 4).

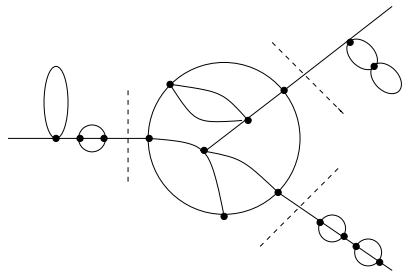


Figure 3. An example of a connected three-point function in the $g\phi^3 + \lambda\phi^4$ scalar field theory

5.2.2 Loop expansion for the effective action

In order to obtain the loop expansion for the effective action we "resuscitate" the constant $\hbar$. We thus have

$$e^{iW[J]/\hbar} = e^{i(\Gamma[\varphi]+J\cdot\varphi)/\hbar} = \int D\tilde{\phi}\; e^{i(S[\tilde{\phi}]+J\cdot\tilde{\phi})/\hbar} \tag{5.73}$$

where, for a reason that will be shortly clear we called $\tilde{\phi}$ the dummy variable. Using that $J^i = -\partial^i\Gamma$ we can rewrite the latter as

$$\begin{aligned} e^{i\Gamma[\varphi]/\hbar} &= \int D\tilde{\phi}\; e^{i(S[\tilde{\phi}]+\partial^i\Gamma(\varphi_i-\tilde{\phi}_i)/\hbar} \\ &= \int D\phi\; e^{\frac{i}{\hbar}S[\phi+\varphi]-\frac{i}{\hbar}\partial^i\Gamma\phi_i} \end{aligned} \tag{5.74}$$

and Taylor expanding the action S about the classical field

$$S[\phi+\varphi] = S[\varphi] + S^i\phi_i + \frac{1}{2}S^{ij}\phi_i\phi_j + \underbrace{\sum_{n\geq 3}\frac{1}{n!}S^{i_1\ldots i_n}\phi_{i_1}\cdots\phi_{i_n}}_{S_{int}[\varphi,\phi]}\ , \tag{5.75}$$

where $S^{i_1\ldots i_n}$ are functional derivatives of S computed in φ, we get

$$e^{\frac{i}{\hbar}(\Gamma[\varphi]-S[\varphi])} = \int D\phi\; e^{\frac{i}{2\hbar}S^{ij}\phi_i\phi_j+\frac{i}{\hbar}S_{int}[\varphi,\phi]-\frac{i}{\hbar}(\partial^i\Gamma-S^i)\phi_i} \tag{5.76}$$

where we already see that, at leading order, the effective action is the classical action. We can thus write their difference as an expansion in $\hbar$

$$\Gamma[\varphi] - S[\varphi] = \sum_{n=1}^{\infty}\hbar^n\Gamma_n[\varphi] \tag{5.77}$$

that determines the quantum corrections to the classical action: the power of $\hbar$ turns out to count the number of loops in the perturbative expansion. Then rescaling the dummy variable as $\phi \to \hbar^{1/2}\phi$ we obtain

$$e^{\underbrace{i\sum_{n\geq 1}\hbar^{n-1}\Gamma_n[\varphi]}_{\text{loop expansion}}} = \int D\phi\; e^{\underbrace{\frac{i}{2}S^{ij}\phi_i\phi_j}_{\text{propagator}}+\underbrace{i\sum_{k\geq 3}\frac{\hbar^{\frac{k}{2}-1}}{k!}S^{i_1\ldots i_k}\phi_{i_1}\cdots\phi_{i_k}}_{\text{vertices}}\underbrace{-i\sum_{n\geq 1}\hbar^{n-\frac{1}{2}}\partial^i\Gamma_n\phi_i}_{\text{vertices to remove non 1PI}}} \tag{5.78}$$

that is our final formula for the loop expansion of the effective action. Very simple turns out to be the one-loop truncation of the previous formula, namely

$$e^{i\Gamma_1[\varphi]} = \int D\phi \; e^{\frac{i}{2}S^{ij}[\varphi]\phi_i\phi_j} = \mathrm{Det}^{-1/2}(-iS^{ij}[\varphi]) \tag{5.79}$$

from which

$$\Gamma[\varphi] = S[\varphi] + \hbar\Gamma_1[\varphi] + O(\hbar^2) \tag{5.80}$$

$$\Gamma_1[\varphi] = \frac{i}{2}\ln\mathrm{Det}(-iS^{ij}[\varphi]) = \frac{i}{2}\mathrm{Tr}\ln(-iS^{ij}[\varphi]) \tag{5.81}$$

that is the famous "trace-log" formula. For constant $\varphi(x) = \varphi_0$ the latter is space-time independent, and turns into a potential

$$\Gamma_1[\varphi_0] = \frac{i}{2}\mathrm{tr}\ln(-iS^{ij}[\varphi_0]) = -U_4 V_{CW}(\varphi_0) \tag{5.82}$$

the Coleman-Weinberg potential; here U_4 is the four-dimensional volume.

In general the one-loop effective action finds a useful representation in terms of relativistic point particle path integrals, as described in section 4.1.2. For the $\lambda\phi^4$ scalar field theory we have

$$S[\varphi] = \frac{1}{2}\int d^4x\varphi(\Box - m^2)\varphi - \frac{\lambda}{4!}\int d^4x\varphi^4 \tag{5.83}$$

$$S^{ij}[\varphi] = \delta^{ij}\Big(\Box - m^2 - \frac{\lambda}{2}\varphi^2(x)\Big) \tag{5.84}$$

so that

$$\Gamma_1[\varphi] = \frac{i}{2}\mathrm{Tr}\ln(-iS^{ij}[\varphi]) = -i\int_0^\infty \frac{dt}{2t}\mathrm{Tr}\; e^{-it(-\Box+M^2(x))} \tag{5.85}$$

where in the second equality we used the Schwinger parametrization of the logarithm of an operator and defined $M^2(x) = m^2 + \frac{\lambda}{2}\varphi^2(x)$. The differential operator $\Box$ can be interpreted as (twice) the hamiltonian of a particle. Hence the latter trace can be represented as a particle path integral with periodic boundary conditions

$$\Gamma_1[\varphi] = -i\int_0^\infty \frac{dt}{2t}\int_{PBC} Dx\; e^{iS[x]} \tag{5.86}$$

with

$$S[x] = \int_0^1 d\tau\Big(\frac{\dot{x}^2}{4t} - tM^2(x(\tau))\Big)\;. \tag{5.87}$$

For $\lambda = 0$ expression (5.86) is nothing but the gauge-fixed circle path integral of a massive relativistic particle. For convergence convenience one often considers the Wick rotated euclidean version of the latter expressions (in such a case $\Box$ turns into a four-dimensional laplacian) i.e.

$$\Gamma_1[\varphi] = -\int_0^\infty \frac{dT}{2T}\int_{PBC} Dx\; e^{-S[x]} \tag{5.88}$$

$$S[x] = \int_0^1 d\tau\Big[\frac{\dot{x}^2}{4T} + TM^2(x(\tau))\Big]\;. \tag{5.89}$$

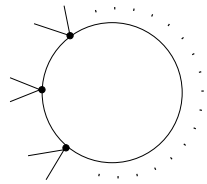


Figure 4. One loop 1PI correlation functions in the $\lambda\phi^4$ theory

The latter simple-looking expression contains a great deal of information about the original scalar field theory. In particular one can extract the one-loop 1PI correlation functions (directly in Fourier space) depicted in Figure 4 in a very simple way with the following recipe:

- write the classical field as a sum of external scalar particle of definite momenta, i.e. $\varphi(x) = \sum_{i=1}^{N} e^{ip_i\cdot x}$, N even;
- expand the exponent and collect all terms multilinear in each $e^{ip_i\cdot x}$;
- compute the particle correlation functions;
- obtain $\Gamma_1^{(N)}(p_1, \ldots, p_N)$.

To conclude this section let us consider a two-loop prove of the 1PI irreducibility of the effective action. To such extent we only need to retain terms up to order $\hbar$ in the expansion (5.78), i.e.

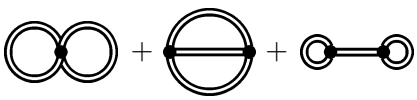
$$e^{i\Gamma_1 + i\hbar\Gamma_2} = \int D\phi \; e^{\frac{i}{2}S^{ij}\phi_i\phi_j} \; e^{\frac{i\hbar^{1/2}}{3!}S^{ijk}\phi_i\phi_j\phi_k + \frac{i\hbar}{4!}S^{ijkl}\phi_i\cdots\phi_l - i\hbar^{1/2}\partial^i\Gamma_1\phi_i} \tag{5.90}$$

that, dividing by $e^{i\Gamma_1}$, reduces to

$$e^{i\hbar\Gamma_2} = \frac{\int D\phi \; e^{\frac{i}{2}S^{ij}\phi_i\phi_j} \; e^{\frac{i\hbar^{1/2}}{3!}S^{ijk}\phi_i\phi_j\phi_k + \frac{i\hbar}{4!}S^{ijkl}\phi_i\cdots\phi_l - i\hbar^{1/2}\partial^i\Gamma_1\phi_i}}{\int D\phi \; e^{\frac{i}{2}S^{ij}\phi_i\phi_j}} \tag{5.91}$$

then expanding both sides we can identify

$$\begin{aligned} i\Gamma_2[\varphi] &= i\frac{1}{4!}S^{ijkl}\langle\phi_i\cdots\phi_l\rangle_\varphi + \frac{1}{2!}\left(\frac{i}{3!}\right)^2 S^{ijk}S^{i'j'k'}\langle\phi_i\phi_j\phi_k\phi_{i'}\phi_{j'}\phi_{k'}\rangle_\varphi \\ &+ \frac{1}{3!}S^{ijk}\partial^{l'}\Gamma_1\langle\phi_i\phi_j\phi_k\phi_{l'}\rangle_\varphi - \frac{1}{2!}\partial^i\Gamma_1\partial^{j'}\Gamma_1\langle\phi_i\phi_{j'}\rangle_\varphi \end{aligned} \tag{5.92}$$

$=$ 

$+$  (5.93)

where

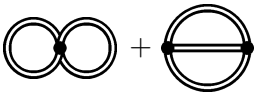
$$\langle \phi_i \phi_{j'} \rangle_\varphi = \frac{\int D\phi \; e^{\frac{i}{2} S^{ij} \phi_i \phi_j} \phi^i \phi^{j'}}{\int D\phi \; e^{\frac{i}{2} S^{ij} \phi_i \phi_j}} = -i S^{-1}_{ij} = \tag{5.94}$$

is the propagator, in the background φ and the star represents the tadpole vertex $\partial^i \Gamma_1$; above (un)primed indices belong to the same vertex. Notice that the tadpole $\partial^i \Gamma_1$ depends on the one-loop effective action that we already computed above. Then some algebra leads to

$$\partial^l \Gamma_1[\varphi] = \frac{1}{2}(-i S^{-1}_{ij}) S^{ijl} \tag{5.95}$$

that inserted into (5.92) determines the cancellation of the non 1PI diagrams, leaving

$$\begin{aligned} \Gamma_2[\varphi] &= \frac{1}{8} S^{ijkl} (-i S^{-1}_{ij})(-i S^{-1}_{kl}) + \frac{i}{12} S^{ijk} S^{i'j'k'} (-i S^{-1}_{ii'})(-i S^{-1}_{jj'})(-i S^{-1}_{kk'}) \\ &= \end{aligned} \tag{5.96}$$



i.e. only 1PI diagrams. The double line notation, as opposed to single line adopted above, is used to emphasize the fact that the propagator is not the simple free propagator but rather a φ-dependent one. A dependence on the background field φ is also present in the vertices $S^{i_1 \dots 1_n}$.

5.2.3 Lehmann-Symanzik-Zimmermann reduction formula

So far we, in the present section, we have considered a somewhat cumbersome mathematical apparatus that appears to have little do we physical (i.e. scattering) processes. The link between scattering amplitudes and correlation function sits in the renowned "Lehmann-Symanzik-Zimmermann reduction formula" that we briefly describe here. The S-matrix element between an initial state $|i\rangle$ and a final state $|f\rangle$ is defined as

$$S_{fi} = \langle f | \mathbb{S} | i \rangle = \langle f | U(\infty, -\infty) | i \rangle \tag{5.97}$$

where, for a scalar field theory, the asymptotic states are characterized only by the momenta of the incoming and outgoing particles

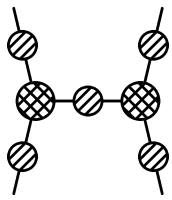
$$|i\rangle = |\mathbf{p}_1, \dots, \mathbf{p}_N\rangle \,, \quad |f\rangle = |\mathbf{p'}_1, \dots, \mathbf{p'}_{N'}\rangle \tag{5.98}$$

for a process with N incoming particles and N' outgoing particles. In the absence of interaction particles "miss each other" and $S_{fi} = \delta_{fi}$. However, in scattering events one is interested in processes where "something happens" and therefore the operator one is interested is $i\mathbb{T} \equiv \mathbb{S} - \mathbb{1}$. In particular we are interested in processes where all the incoming particle interact with each other that from diagrammatic point view means we only need to

consider "connected diagrams". The Lehmann-Symanzik-Zimmermann reduction formula precisely relates matrix elements of operator $i\mathbb{T}$ to connected correlation functions, namely

$$\begin{aligned}\langle \mathbf{p}'_1, \ldots, \mathbf{p}'_{N'} | i\mathbb{T} | \mathbf{p}_1, \ldots, \mathbf{p}_N \rangle &= i(2\pi)^4 \delta^{(4)}\Big(\sum_i p_i - \sum_{i'} p'_{i'}\Big)\mathcal{M}(\{p_i\} \to \{p'_{i'}\}) \\ &= \frac{\widetilde{W}^{(N+N')}(\{p_i\}, \{p'_{i'}\})}{\prod_i \widetilde{W}^{(2)}(p_i) \prod_{i'} \widetilde{W}^{(2)}(p'_{i'})} \end{aligned} \tag{5.99}$$

where $\widetilde{W}$ are connected correlation functions in momentum space. The quotient above is know under the name of "truncation" as diagrammatically it corresponds to the removal of all the external two-point functions. For the $1 \to 2$ process of Figure 3 it precisely corresponds to removing the part of diagrams outside the dashed lines. Hence, for a process involving a total of three particles ($1 \to 2$ or $2 \to 1$) the Feynman scattering amplitude $\mathcal{M}$ is just given by the 1PI three-point function. For a higher order process such as the $2 \to 2$ s-channel described by



(5.100)

the scattering amplitude involves the two 1PI three-point functions connected with the internal $W^{(2)}$. Let us conclude this section by mentioning that the material proposed here can be mostly found in the book [14] in Chapters 4 and 9.

5.3 Problems for Section 5

(1) Show the second passage of equation (5.9) by showing that it satisfies $i\frac{d}{dt_2}U(t_2, t_1) = \mathbb{H}_I(t_2)U(t_2, t_1)$ and $U(t_2, t_2) = \mathbb{1}$.

(2) Find the functional and diagrammatic relations between $G^{(4)}$ and $W^{(k)}$, $k \leq 4$.

(3) By taking a further derivative of expression (5.68) obtain irreducibility condition (5.72).

(4) Diagrammatically represent (5.72).

(5) Complete the passages that lead to the cancellation of non 1PI diagrams in (5.96).

Acknowledgments

The author acknowledges UCMEXUS-CONACYT Collaborative Grant CN-12-564 for financial support on the carrying out of the "School on Spinning Particles in Quantum Field Theory: Worldline Formalism, Higher Spins and Conformal Geometry," held in Morelia (Michoacán) Mexico, 19-23 November 2012.

6 Final Comments

What included in the present manuscript is an expanded version of a set of lectures given in Morelia (Mexico) in November 2012 and in the Physics Department of the University of Modena in January 2013, where we reviewed some elementary material on particle path integrals and introduced a list of (more or less) recent topics where particle path integrals can be efficiently applied. In particular we pointed out the role that some locally-symmetric relativistic particle models have in the first-quantized description of higher spin fields [15]. However the list of topics reviewed here is by no means complete. Firstly, coupling to external gravity has been overlooked (it has been partly covered in [11].) In fact, path integrals in curved spaces involve nonlinear sigma models and perturbative calculations have to be done carefully as superficial divergences appear. Although this issue is well studied and understood by now, it had been source of controversy (and errors) in the past (see [16] for a review.) Moreover, for the $O(N)$ spinning particle models discussed above, coupling to gravity is not straightforward as (for generic N) the background appears to be constrained [7] and the symmetry algebra is not linear and thus the topic is an on-going research argument (see [12] for some recent results.)

Some other modern applications of particle path integrals that have not been covered here, include: the numerical worldline approach to the Casimir effect [17], AdS/CFT correspondence and string dualities [18], photon-graviton amplitudes computations [19], nonperturbative worldline methods in QFT [5, 20], QFT in curved space time [21], the worldgraph approach to QFT [9], as well as the worldline approach to QFT on manifolds with boundary [22] and to noncommutative QFT [23].

A Natural Units

In quantum field theory it is often convenient to use so called "natural units": for a generic physical quantity X, its physical units can always be express in terms of energy, angular-momentum $\times$ velocity and angular momentum as

$$[X] = E^a (Lc)^b L^c = m^\alpha l^\beta t^\gamma \tag{A.1}$$

with

$$\begin{aligned} a &= \alpha - \beta - \gamma \\ b &= \beta - 2\alpha \\ c &= \gamma + 2\alpha \ . \end{aligned} \tag{A.2}$$

Therefore, if velocities are measured in units of the speed of light c and angular momenta are measured in units of the Planck constant $\hbar$, and thus are numbers, we have the natural units for X given by

$$[X]\big|_{n.u.} = E^{\alpha-\beta-\gamma} \,, \quad (\hbar c)\big|_{n.u.} = 1 \,, \quad \hbar\big|_{n.u.} = 1 \tag{A.3}$$

and energy is normally given in MeV. Conversion to standard units is then easily obtained by using

$$\hbar c = 1.97 \times 10^{-11} \text{ MeV} \cdot \text{cm} \,, \quad \hbar = 6.58 \times 10^{-22} \text{ MeV} \cdot \text{s} \ . \tag{A.4}$$

For example a distance expressed in natural units by $d = 1$ (MeV)$^{-1}$, corresponds to $d = 1.97 \times 10^{-11}$ cm.

Another set of units, very useful in general relativity, makes use of the Newton constant G (divided by c^2) and the speed of light. The first in fact has dimension of inverse mass times length. For a generic quantity we thus have

$$[X] = l^a v^b (m^{-1} l)^c = m^\alpha l^\beta t^\gamma \tag{A.5}$$

with

$$\begin{aligned} a &= \alpha + \beta + \gamma \\ b &= -\gamma \\ c &= -\alpha \ . \end{aligned} \tag{A.6}$$

Therefore, if velocities are measured in units of the speed of light c and $m^{-1}l$ measured in units of G/c^2, and thus are numbers, we have the geometric units for X given by

$$[X]\big|_{g.u.} = m^{\alpha+\beta+\gamma} \ , \quad (c)\big|_{g.u.} = 1 \ , \quad \frac{G}{c^2}\big|_{g.u.} = 1 \tag{A.7}$$

and length is normally given in (centi)meters. Conversion to standard units is then easily obtained by using

$$c\big|_{i.s.} = 3 \times 10^8 \text{ m s}^{-1} \ , \quad \frac{G}{c^2}\big|_{i.s.} = 7.425 \times 10^{-28} \text{ m kg}^{-1} \ . \tag{A.8}$$

Then for an arbitrary quantity we have

$$X\big|_{i.s.} = \underbrace{x|_{g.u.} c^{-\gamma} \left(\frac{G}{c^2}\right)^{-\alpha}}_{x|_{i.s.}} \text{ kg}^\alpha \text{ m}^\beta \text{ s}^\gamma \tag{A.9}$$

where x's are numbers. For example, for the electron mass we have

$$m_e|_{i.s.} = 9.11 \times 10^{-31} \text{ kg} \ , \qquad \underbrace{\longrightarrow}_{\alpha=1,\, \beta=\gamma=0} \qquad m_e|_{g.u.} = 6.76 \times 10^{-58} \text{ m} \ . \tag{A.10}$$

The mass in geometric units represents (half of) the Schwarzschild radius of the particle, r_S. For astronomical objects it is a quite important quantity as it represents the event horizon in the Schwarzschild metric (for the sun we have $2M \approx 3$ km). Such horizon does not depend on the mass density only on the mass; on the other hand the physical radius does depend on the mass density. Then, if $r_S \geq R_{phys}$ the object is very dense and the horizon appears outside the physical radius where the Schwarzschild is valid. In that case only the spherical object gives rise to a horizon and the object is a "black hole".

To conclude let us just mention another set of units, the Planck units, where $G = c = \hbar = (4\pi\epsilon_0)^{-1} = k_B = 1$, that is, all physical quantities are represented by numbers.

B Action principle and functional derivative

We briefly review the action principle for a particle as a way to introduce functional derivatives. A particle on a line is described by the action

$$S[x] = \int_0^t d\tau \; L(x, \dot{x}) \tag{B.1}$$

with $x(\tau)$ describing the "path" of the particle from time 0 to time t. The classical path with fixed boundary conditions $x(0) = x'$ and $x(t) = x$ is the one that stabilizes the functional. Considering a small variation $\delta x(\tau)$, centered on an arbitrary path $x(\tau)$ with the above b.c.'s, so that $\delta x(0) = \delta x(t) = 0$, we have

$$\delta S[x] = \int_0^t d\tau \; \delta x(\tau) \left(\frac{\partial L}{\partial x} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}} \right) \tag{B.2}$$

as linear variation of the action, that can be interpreted as functional differential of the action functional, in strict analogy with multivariable differentials of a function. The functional derivative of the action functional can thus be defined as

$$\frac{\delta S[x]}{\delta x(\tau)} = \frac{\partial L}{\partial x} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}} \tag{B.3}$$

and the Euler-Lagrange equation thus follows from the vanishing of the functional derivative. That is we can treat $x(\tau)$ as a set of independent real variables, with a continuum index τ as use the analogy with multivariable calculus to define the rules for the functional derivative, that are as follows

1. $\frac{\delta x(\tau^i)}{\delta x(\tau^j)} = \delta(\tau^i - \tau^j)$, generalization of $\frac{\partial x^i}{\partial x^j} = \delta^i_j$;
2. $\frac{\delta}{\delta x(\tau)} \Big(ST \Big) = \frac{\delta S}{\delta x(\tau)} T + S \frac{\delta T}{\delta x(\tau)}$, the Leibnitz rule;
3. is $y(\tau)$ can be seen as a functional of τ we can use the "functional chain rule", $\frac{\delta S[y]}{\delta x(\tau)} = \int d\tau' \frac{\delta S[y]}{\delta y(\tau')} \frac{\delta y(\tau')}{\delta x(\tau)}$.

The above analogy also allows to use indexed variables x^i as shortcuts for functions $x(\tau^i)$ and (implied) sums $\sum_i$ as shortcut for integrals $\int d\tau^i$. In section 5 a similar shortcut is used for four-dimensional functions (fields), integrals and derivatives.

C Fermionic coherent states

This is a compendium of properties of fermionic coherent states. Further details can be found for instance in [24].

The even-dimensional Clifford algebra

$$\{\psi^M, \psi^N\} = \delta^{MN} \,, \quad M, N = 1, \ldots, 2l \tag{C.1}$$

can be written as a set of l fermionic harmonic oscillators (the index M may collectively denote a set of indices that may involve internal indices as well as a space-time index), by simply taking complex combinations of the previous operators

$$a^m = \frac{1}{\sqrt{2}}\left(\psi^m + i\psi^{m+l}\right) \tag{C.2}$$

$$a^\dagger_m = \frac{1}{\sqrt{2}}\left(\psi^m - i\psi^{m+l}\right) , \qquad m = 1, \ldots, l \tag{C.3}$$

$$\{a^m, a^\dagger_n\} = \delta^m_n \tag{C.4}$$

and it can be thus represented in the vector space spanned by the 2^l orthonormal states $|\mathbf{k}\rangle = \prod_m (a^\dagger_m)^{k_m}|0\rangle$ with $a_m|0\rangle = 0$ and the vector $\mathbf{k}$ has elements taking only two possible values, $k_m = 0, 1$. This basis (often called spin-basis) yields a standard representation of the Clifford algebra, i.e. of the Dirac gamma matrices.

An alternative overcomplete basis is given by the coherent states that are eigenstates of creation or annihilation operators

$$|\xi\rangle = e^{a^\dagger_m \xi^m}|0\rangle \quad \rightarrow \quad a^m|\xi\rangle = \xi^m|\xi\rangle = |\xi\rangle\xi^m \tag{C.5}$$

$$\langle\bar\eta| = \langle 0|e^{\bar\eta_m a^m} \quad \rightarrow \quad \langle\bar\eta|a^\dagger_m = \langle\bar\eta|\bar\eta_m = \bar\eta_m\langle\bar\eta| \ . \tag{C.6}$$

Below we list some of the useful properties satisfied by these states. Using the Baker-Campbell-Hausdorff formula $e^X e^Y = e^Y e^X e^{[X,Y]}$, valid if $[X, Y] = c$-number, one finds

$$\langle\bar\eta|\xi\rangle = e^{\bar\eta\cdot\xi} \tag{C.7}$$

that in turn implies

$$\langle\bar\eta|a^m|\xi\rangle = \xi^m\langle\bar\eta|\xi\rangle = \frac{\partial}{\partial\bar\eta_m}\langle\bar\eta|\xi\rangle \tag{C.8}$$

$$\langle\bar\eta|a^\dagger_m|\xi\rangle = \bar\eta_m\langle\bar\eta|\xi\rangle \tag{C.9}$$

so that $\{\frac{\partial}{\partial\bar\eta_m}, \bar\eta_n\} = \delta^m_n$. Defining

$$d\bar\eta = d\bar\eta_l \cdots d\bar\eta_1 \ , \qquad d\xi = d\xi^1 \cdots d\xi^l \tag{C.10}$$

so that $d\bar\eta d\xi = d\bar\eta_1 d\xi^1 d\bar\eta_2 d\xi^2 \cdots d\bar\eta_l d\xi^l$, one finds the following relations

$$\int d\bar\eta d\xi \ e^{-\bar\eta\cdot\xi} = 1 \tag{C.11}$$

$$\int d\bar\eta d\xi \ e^{-\bar\eta\cdot\xi} \ |\xi\rangle\langle\bar\eta| = \mathbb{1} \tag{C.12}$$

where $\mathbb{1}$ is the identity in the Fock space. One can also define a fermionic delta function with respect to the measure (C.10) by

$$\delta(\bar\eta - \bar\lambda) \equiv (\bar\eta^1 - \bar\lambda^1)\cdots(\bar\eta^l - \bar\lambda^l) = \int d\xi \ e^{(\bar\lambda - \bar\eta)\cdot\xi} \ . \tag{C.13}$$

Finally, the trace of an arbitrary operator can be written as

$$\mathrm{Tr}\, A = \int d\bar{\eta} d\xi \; e^{-\bar{\eta}\cdot\xi} \langle -\bar{\eta}|A|\xi\rangle = \int d\xi d\bar{\eta} \; e^{\bar{\eta}\cdot\xi} \langle \bar{\eta}|A|\xi\rangle \; . \tag{C.14}$$

As a check one may compute the trace of the identity

$$\mathrm{Tr}\, \mathbb{1} = \int d\xi d\bar{\eta} \; e^{\bar{\eta}\cdot\xi} \langle \bar{\eta}|\xi\rangle = \int d\xi d\bar{\eta} \; e^{2\bar{\eta}\cdot\xi} = 2^l \; . \tag{C.15}$$

D Noether theorem

The Noether theorem is a powerful tool that relates continuous global symmetries to conserved charges. Let us consider a particle action $S[x]$: dynamics associated to this action is the same as that obtained from the action $S'[x]$ that differs from $S[x]$ by a boundary term, i.e. $S'[x] \cong S[x]$. Then, if a continuous transformation of fields $\delta x^i = \alpha \Delta x^i$, parametrized by a continuous parameter α, yields $S[x + \alpha\Delta x] = S'[x]\,,\ \forall x^i(\tau)$, the action is said to be symmetric upon that transformation. Moreover, taking the parameter to be time-dependent the variation of the action will be given by

$$\delta S \equiv S[x + \alpha\Delta x] - S'[x] = \int d\tau \; \dot{\alpha} Q \; . \tag{D.1}$$

The latter tells us that if we have a continuous symmetry then

$$\dot{\alpha} = 0 \quad \Rightarrow \quad \delta S = 0 \; . \tag{D.2}$$

Moreover, for $\alpha(\tau)$ and for an arbitrary trajectory $x(\tau)$, the variation of the action does not vanish. However, it does vanish ($\forall \alpha(\tau)$) on the mass shell of the particle, i.e. imposing equations of motion. Hence

$$\frac{d}{d\tau} Q\Big|_{on-shell} = 0 \tag{D.3}$$

in other words, $Q\big|_{on-shell}$ is a conserved quantity. The formal construction that lead to (D.1), i.e. the use of a time-dependent parameter, is often called the "Noether trick". The just-described Noether theorem holds at the classical level. Below we describe how it gets modified if we consider a quantum version of the classical particle system considered here. However let us first consider an example at the classical level; let us take the non-relativistic free particle, described by

$$S[x] = \frac{m}{2} \int_0^t d\tau \; \dot{x}^2 \,, \quad x = (x^1, \ldots, x^d) \tag{D.4}$$

for which the on-shell condition is obviously $\ddot{x}^i = 0$. A set of continuous symmetry for which the latter is invariant is:

1. time translation: $\delta\tau = -a\,, \quad \delta x^i = a\dot{x}^i$;
2. spatial translations: $\delta x^i = a^i$;

3. rotations: $\delta x^i = \theta^{ij} x^j$, with $\theta^{ij} = -\theta^{ji}$;

4. Galilean boosts: $\delta x^i = -v^i \tau$;

with time-independent infinitesimal parameters a, a^i, θ^{ij}, v^i. In order to check the invariance and obtain the conserved charges with turn the parameters into time-dependent ones $a(\tau), a^i(\tau), \theta^{ij}(\tau), v^i(\tau)$ and consider the variation of the above action. We get

1. $\delta S = \int_0^t d\tau \ \dot{a} E$, $\longrightarrow$ conservation of energy $E = \frac{m}{2}\dot{x}^2$;

2. $\delta S = \int_0^t d\tau \ \dot{a}^i P^i$, $\longrightarrow$ conservation of momentum $P^i = m\dot{x}^i$;

3. $\delta S = \int_0^t d\tau \ \dot{\theta}^{ij} J^{ij}$, $\longrightarrow$ conservation of angular momentum $J^{ij} = \frac{m}{2}(\dot{x}^i x^j - x^i \dot{x}^j)$;

4. $\delta S = \int_0^t d\tau \ \dot{v}^i X^i$, $\longrightarrow$ conservation of center of mass motion $X^i = x^i - \frac{1}{m} P^i \tau$.

The result is thus two-fold. On the one hand, for time-independent parameters the action is invariant off-shell and correspondingly, for time-dependent parameters its variation is proportional to the conserved charge. In fact on-shell $\delta S = 0$ and by partial integration one gets the conservation laws $\dot{E}|_{o.s.} = \dot{P}^i|_{o.s.} = \dot{J}^{ij}|_{o.s.} = \dot{X}^i|_{o.s.} = 0$. The latter can be easily checked to hold.

The quantum version of the Noether can be for instance obtained through the path integral approach. In such a case we deal with sourceful functionals like

$$F[j] = \int Dx \ e^{iS[x] + i\int jx} = \int D\bar{x} \ e^{iS[\bar{x}] + i\int j\bar{x}} \tag{D.5}$$

where in the second equality we just renamed the dummy variable –that is what we may call the "Shakespeare" theorem (what's in a name?). Now, let us take $\bar{x} = x + \delta_\alpha x$ where $\delta_\alpha x = \alpha \Delta x$ is an infinitesimal transformation of the coordinate x parametrized by a continuous parameter α, as above. If such transformation is a (classical) symmetry, the Noether theorem implies that $S[\bar{x}] = S[x] + \int \dot{\alpha} Q$, with α made time-dependent and Q being the associated classically-conserved charge. Then to linearized level in α we have

$$F[j] = \int Dx \ J_\alpha \ e^{iS[x] + i\int jx} \Big[1 + i \int d\tau \alpha \big(j\Delta x - \dot{Q} \big) \Big] \tag{D.6}$$

after partial integration. The jacobian term J_α comes from the change of integration variable, from $\bar{x} = x + \alpha \Delta x$ to x. If the Jacobian is equal one, $F[j]$ also appears on the r.h.s, so that subtracting it from both sides leaves

$$\int Dx \ e^{iS[x] + i\int jx} \Big[j(\tau)\Delta x(\tau) - \dot{Q}(\tau) \Big] = 0 \tag{D.7}$$

the is the quantum counterpart of the classical Noether theorem. In particular setting $j = 0$ one gets

$$\int Dx \ e^{iS[x]} \dot{Q} = 0 \tag{D.8}$$

that is the quantum-mechanical conservation of the charge Q. However, notice that (D.7) contains a lot more information then its classical counterpart. In fact one can differentiate (D.7) w.r.t. j an arbitrary number of times and set $j = 0$ in the end. This leads to an infinite tower of identities called "Ward identities". For example if we take one derivative w.r.t. $j(\tau')$ and set $j = 0$ we have

$$\int Dx\ e^{iS[x]+i\int jx}\Big[\delta(\tau-\tau')\Delta x(\tau) - i\dot{Q}(\tau)x(\tau')\Big] = 0 \tag{D.9}$$

and similarly for higher order differentiations. On the other hand if the Jacobian is NOT equal to one, then the classical symmetry does not hold at the quantum level, i.e. $\int Dx\ e^{iS[x]}\dot{Q} = \mathcal{A} \neq 0$. The quantity $\mathcal{A}$ is called "anomaly". Notice that, since we are dealing with infinitesimal transformations, we always (schematically) have $J_\alpha = 1 + i\alpha\mathcal{J}$. The anomaly is then precisely the path integral average of $\mathcal{J}$, in other words the anomaly sits in the path integral Jacobian [25]. For the free particle example considered above it is easy to see that the Jacobian is identically one. Let us check it for instance for the rotations: we have

$$J_\theta = \det\frac{\partial\bar{x}^{i'}(\tau')}{\partial x^i(\tau)} = \det\big(\delta^{i'i}\delta(\tau-\tau') + \theta^{i'i}\delta(\tau-\tau')\big) = 1 + \mathrm{tr}\big(\theta^{i'i}\delta(\tau-\tau')\big) = 0 \tag{D.10}$$

where, in the second equality we used that $\det(1+A) = 1+\mathrm{tr}A+o(A^2)$ if A is infinitesimal, whereas the third equality follows from the antisymmetry of θ. Similarly one can show that the other three symmetries considered above give rise to a unit Jacobian.

What described in the present section for a particle generalizes to quantum field theories. In particular let us conclude mentioning that (cancellation of) anomalies have played a crucial role in theoretical physics in the past decades to describe or predict several physical processes, the π^0 decay, the Aharonov-Bohm effect and the quark top prediction just to name a few.

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