Problems

Problems marked with * are meant to be solved without the aid of a computer.

(1) Code the Monte Carlo solution for the area of a circle to find π. Extend it to 3 dimensions to get π from the volume of a sphere $\frac{4\pi}{3}$. If you use RAN3 call it several times between usages for the second part of this problem to avoid the correlations among 3 successive values.

(2) Continue the previous problem by increasing the dimension up to 10. Observe that the fractional error of the result deteriorates rapidly with increasing dimension.

(3) Integrate x, x^2, x^3, x^4 and x^5 from 0 to 1 using Monte Carlo with x chosen uniformly from 0 to 1.

(4) Generate distributions for $h_n(x) = (n+1)x^n \quad 0 \le x \le 1$, put them in bins and compare with the expected *pdf* for $n = 1,\ 2$ and 3.

(5) Perform the integral:

$$\int_0^1 x^5 dx = \frac{1}{n+1}\int_0^1 h_n(x) x^{5-n} dx$$

for $n = 0,\ 1,\ 2,\ 3,\ 4,\ 5$ by Monte Carlo. Compare the relative accuracy by examining the variance computed from Eq. 2.34

(6) * Write an equation to sample x from the normalized probability distribution function

$$g(x) = \frac{3x^2}{(1+x^3)^2} \qquad 0 \le x \le \infty.$$

Note that

$$\int_0^x g(x)dx = \frac{x^3}{1+x^3}.$$

(7) * Derive the normalized form, the cumulative distribution function, $F(x)$, and the sampling equation for:

$$g_3: \quad \int \frac{x dx}{(a^2+x^2)^{n+1}} = -\frac{1}{2n(a^2+x^2)^n}$$

and

$$g_4: \quad \int \frac{dx}{(a^2+b^2x)x^{\frac{1}{2}}} = \frac{2}{ab}\tan^{-1}\frac{bx^{\frac{1}{2}}}{a}.$$

(8) Write a function type subroutine to sample g_5, i.e.

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x=g5(an,a)
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which produces a value drawn from the distribution g_5 in Table 2.1. Write and execute a test program which calls this function, bins the results and compares with the analytic expression for g_5.

(9) Write a code to sample the function

$$\left(\frac{r}{\sqrt{a^2+r^2}}\right)^{n+1} b e^{-br} \qquad 0 \le r \le \infty$$

using the method of Section 2.3.9. Make use of the subroutine to sample g5(r) that you created in the previous problem. Take $a = b = n = 1$ so that the function behaves as r^2 near the origin. Bin your results and compare with the expected distribution.

(10) * Prepare to sample a probability distribution function proportional to

$$\frac{1}{x}; \qquad 1 \le x \le a.$$

Find the normalized *pdf*, the cumulative distribution function and the sampling equation.

(11) Use the sum of $x_1 + x_2 + x_3$ to sample

$$(e^{-ax} - e^{-bx})^2$$

by taking:

$$x_1 \text{ from } 2ae^{-2ax}$$

$$x_2 \text{ from } 2be^{-2bx}$$

$$x_3 \text{ from } (a+b)e^{-(a+b)x}.$$

Bin and compare with the exact distribution. Use RAN3 and the normal random number generator on your machine. Note that the results with RAN3 are very bad for small x. Use $a = 0.5,\ b = 1.0$.

(12) Use the selection method (Section 2.3.9) to evaluate

$$\int_0^\infty e^{-x}\ \mathrm{erf}(x)dx \quad [= e^{\frac{1}{4}}\ \mathrm{erf}(\frac{1}{2})]$$

from the ratio of Monte Carlo throws to the total number of trials.

(13) Calculate

$$\int_0^\infty \frac{e^{-at}dt}{\sqrt{t}(t+z)} = \frac{\pi}{\sqrt{z}}e^{az}\mathrm{erfc}\sqrt{az}$$

and

$$\int_0^\infty e^{-(at^2+2bt)}dt = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{a}}\mathrm{erfc}\frac{b}{\sqrt{a}}$$

using the "product of functions" method of sampling discussed in section 2.3.9 for a, b, and $z = 1$.

(14) *For the normalized *pdf*

$$g(x) = a + bx; \quad 0 \le x \le 1; \quad a \ge 0; \quad b \ge 0; \quad a + \frac{1}{2}b = 1:$$

a) Find the *cdf* and the sampling equation for direct sampling.
b) Prepare the function to allow for sampling by the method of the sum of two *pdf*s. Sketch the coding for doing this.
c) Prepare the function for sampling by the rejection method. Sketch the coding for doing this.

(15) *Consider the *pdf* in the range $0 \le x \le 1$

$$g(x) = nax^{n-1} + 2nbx^{2n-1}; \quad a > 0; \quad b > 0: \quad a + b = 1$$

a) Find the *cdf*, F(x). Derive the sampling equation and show that it has the correct limits for $F = 0$ and $F = 1$.
b) Prepare the function to be sampled by the rejection technique.
c) Sketch the technique for sampling this distribution as the sum of two *pdf*s, assuming that you know how to sample any power of x.

(16) *Consider a Metropolis problem in a discrete variable x. Follow a single walker which starts at $x = 0$. Use the following values for the *pdf*, $g(x)$.

$$g(-2) = 0$$
$$g(-1) = \tfrac{1}{4}$$
$$g(0) = \tfrac{1}{4}$$
$$g(1) = \tfrac{1}{2}$$
$$g(2) = 0$$

Assume the following conditions and fill in the last column of the array.

Step Number	Proposed Step	Random Number	Position
1	+1	0.634	
2	+1	0.216	
3	−1	0.352	
4	+1	0.967	
5	−1	0.448	
6	−1	0.343	
7	+1	0.121	
8	−1	0.749	
9	+1	0.691	
10	+1	0.742	

Calculate the average value of x and x^2 and compare with the exact result. Do not include the starting position in your calculation.

(17) Use the Metropolis algorithm to sample $2x$ in the range $0 \le x \le 1$. Start from a uniform distribution and use 1000 walkers.
i) Propose a step uniformly in a range ±0.1 about the current x. Check to make sure that the proposed x does not fall outside the range $0 \to 1$.
To show the more general possibility for choosing the new position
ii) Propose the next x in the entire range $0 \to 1$.
In both cases bin the results and print out, side-by-side, x, the exact answer, the initial distribution and the result after each 10 time steps. Use 20 bins and a total of 100 time steps. Note that the exact answer for the fraction of the total number of walkers in a bin is $x^2_{max} - x^2_{min}$ where $x_{min(max)}$ is the minimum (maximum) value of x of the bin.

(18) Use the Metropolis algorithm to sample:

$$(x-y)^2 \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Propose a new value of x and y over the entire range. Bin the number of hits in a two-dimensional bin array (integer) and print out the array after every 10 steps. Use 1000 walkers.